# Chapter 2. Normed Linear Spaces: The Basics

## Study Guide

The following is a brief list of topics covered in Chapter 2 of Promislow's *A First Course in Functional Analysis*. This list is not meant to be comprehensive, but only gives a list of several important topics. You should also carefully study the proofs given in class and the homework problems.

**Section 2.1. Metric Spaces.**
Definition of metric space, Triangle Inequality, distance from a point to a set, convergence sequence, continuous functions, diameter of a set, bounded set.

**Section 2.2. Norms.**
Definition of norm, normed linear space, examples of normed linear spaces, the Backwards Triangle Inequality, Continuity of Operations, uniqueness of limits, open ball, closed ball, interior point, exterior point, boundary point, open set, closed set, continuous function at a point and on a set, uniformly continuous function, closure of set $A$ and different ways to express it, dense set, two definitions of compact sets (involving open covers and sequences), compact sets are closed and bounded ("The Compact Set Theorem").

**Section 2.3. Spaces of Bounded Functions.**
Sup norm of a function, bounded functions on set $S$ form a normed linear space $B(S)$, sup norm convergence implies uniform convergence.

**Section 2.4. Bounded Linear Operators.**
Definition of the operator norm on $T \in (X, Y)$, bounded operator, isometry, continuous operator, uniformly continuous operator, boundedness and continuity (Theorem 2.6), multiplication operator, composition of operators.

**Section 2.5. Completeness.**
Cauchy sequences, complete normed linear space (Banach space), properties of Cauchy sequences (Proposition 2.9), fast Cauchy sequence and rapidly Cauchy sequence, series in a normed linear space (convergent, partial sums, absolutely convergent), completeness in terms of absolute convergence (Theorem 2.12), the space of bounded functions is complete (Theorem 2.14), closed subspaces of Banach spaces are Banach spaces (Theorem 2.16), the Extension Theorem (Theorem 2.20), completion of a normed linear space, the Completion Theorem (Theorem 2.22).

**Section 2.6. Comparison of Norms.**
Stronger and weaker norms, relations between norms (Proposition 2,23, 2,24, and Corollary 2.25), equivalent norms, equivalent norms on Banach spaces (Theorem 2.26).

**Section 2.7. Quotient Spaces.**
Equivalence relation, equivalence classes and linear manifolds, definition of quotient space $X/N$ where $X$ is a linear space and $X$ is a subspace, canonical projection concerning quotient spaces, properties of norms on quotient spaces (Theorem 2.27).

**Section 2.8. Finite-Dimensional Normed Linear Spaces.**
$B(F)$ is the set of bounded functions on set $F$, closed and bounded subsets of $B(F)$ are compact (Theorem 2.29), properties of finite dimensional spaces (Theorem 2.31), a perpendicular to a subspace of a normed linear space, Reisz's Lemma (Theorem 2.33; and "near perpendiculars"), classification of finite dimensional normed linear spaces (Riesz's Theorem, Theorem 2.34).

**Section 2.9. $L^p$ Spaces.**
Definition of a $\sigma$-algebra of sets, Borel sets, a measure on a $\sigma$-algebra, measure space, measurable function, finite measrue space, simple function, integral, Lebesgue integral, null set, almost everywhere, properties of integrals of nonnegative measurable functions, Fatou's Lemma, Monotone Convergence Theorem, positive and negative parts of measurable functions, integral of general measurable function, properties of Lebesgue integral of general measurable functions, $\mathcal{L}^p$ spaces for $1 \le p \le \infty$, Hölder's Inequality (Theorem 2.37), Minkowski's Inequality (Theorem 2.38), essential supremum and essentially bounded measurable function, the $L^p$ spaces are complete (The Riesz-Fischer Theorem, Theorem 2.41), the $\ell^p$ spaces for $1 \le p \le \infty$, separable normed linear spaces.

**Section 2.10. Direct Sums and Products.**
Indexing set, product spaces, direct product of normed linear spaces, natural projection, direct sum.

**Section 2.11. Schauder Bases.**
Definition Schauder basis, Hamel basis.

**Supplement. Groups, Fields, and Vector Spaces.**
Definition of group, field, vector space, linear combination, linearly independent set of vectors, span, basis, finite dimensional, solutions of homogeneous systems of equations (lemma 5.1.1), bases are the same size (Theorem 5.1.1), dimension of finite dimensional vector space, isomorphic vector spaces, the Fundamental Theorem of Finite Dimensional Vector Spaces (Theorem 5.1.2), linear transformation, standard basis, matrices and linear transformations (Theorem 5.1.3), equivalence relation, partial ordering, total ordering, maximal element, upper bound, Zorn's Lemma, every vector space has a Hamel basis (Theorem 5.1.4), any two Hamel bases for a given vector space have the same cardinality (Exercise 5.1.3), Schauder basis.

**Section 2.12. Fixed Points and Contraction Mappings.**
Definition of fixed point, contraction mapping, Contraction Mapping Theorem (Theorem 2.44).

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