

# Chapter 4. Hilbert Spaces

## Study Guide

The following is a brief list of topics covered in Chapter 4 of Promislow's *A First Course in Functional Analysis*. This list is not meant to be comprehensive, but only gives a list of several important topics. You should also carefully study the proofs given in class and the homework problems.

### Section 4.1. Introduction.

Dot products.

### Section 4.2. Semi-Inner Products.

Definition of semi-inner product, inner product, semi-norm, the Basic Identity (Lemma 4.2), the Cauchy-Schwarz Inequality (Theorem 4.3), Triangle Inequality (Theorem 4.4), the Parallelogram Law (Proposition 4.5), the Polarization Identity (Proposition 4.7), Continuity of Inner Product (Theorem 4.9), Hilbert space.

### Supplement. Inner Product Spaces.

Definition of inner product space, definition of norm, Schwarz's Inequality (Theorem 5.2.1), The Triangle Inequality (Theorem 5.2.2), orthogonal vectors, orthonormal vectors, normalized vector, The Pythagorean Theorem (Theorem 5.2.3), Bessel's Inequality (Corollary 5.2.1), definition of completeness in an inner product space, Hilbert space.

### Section 4.3. Nearest Points and Convexity.

Definition of convex set, strictly convex space, uniformly convex linear space, any Hilbert space is uniformly convex (Example 4.11), nearest point to a closed convex set (Theorem 4.12), definition of the projection of a point onto a closed subspace.

### Section 4.4. Orthogonality.

Orthogonal vectors, perp set (Proposition 4.13), the Projection Theorem (Theorem 4.14), definition of an orthonormal basis for a Hilbert space, Properties of Orthonormal Sets (Theorem 4.17), every Hilbert space has an orthonormal basis (Theorem 4.18), a separable infinite dimensional Hilbert space is isometric to  $\ell^2$  (Theorem 4.19), the Gram-Schmidt Orthogonal Process (Theorem 4.20),

### Supplement. The Space $L^2$ .

Definitions of  $L^2([a, b])$ ,  $\ell^2(\mathbb{R})$ , and  $\ell^2(\mathbb{C})$ .

**Supplement. The Projections and Hilbert Space Isomorphisms.**

Definition of projection of one vector on another, orthogonal vectors, orthogonal complement of a set,  $S^\perp$  is a Hilbert space (Theorem 5.4.1), nearest point to a subspace (Theorem 5.4.2), direct sums and Theorem 5.4.3, Riesz basis, Schauder basis and orthonormal basis (Theorem 5.4.4), representation in terms of an orthonormal basis (Theorem 5.4.5), the norm of a vector in terms of basis elements (Theorem 5.4.7), separable Hilbert space, Hilbert space isomorphism, Fundamental Theorem of Infinite Dimensional Vector Spaces (Theorem 5.4.9), representation of a linear transformation as an infinite matrix (Theorem 5.4.10).

**Section 4.5. Linear Functionals on Hilbert Spaces.**

Definition of a bounded linear functional, dual space, Theorem 4.22 (and its proof in the supplement),  $H$  and  $H^*$  are the same dimension (Corollary 4.23), dual basis of  $H^*$ .

**Section 4.6. Linear Operators on Hilbert Spaces.**

Definition of sesquilinear form, quadratic form, norm and boundedness of a sesquilinear form, sesquilinear forms correspond to bounded linear transformations (Theorem 4.24), adjoint operator, Properties of Hilbert Space Adjoints (Theorem 4.26), relationships between null spaces and ranges (Proposition 4.27), support of a bounded linear operator, definition of a normal operator and Proposition 4.30, definition of self adjoint operator and Proposition 4.31, definition of a positive operator, square root of a positive operator (Theorem 4.32), definition of projection operator and Proposition 4.33, definition of a unitary operator and Proposition 4.34.

**Section 4.7. Order Relation on Self-Adjoint Operators.**

Definition of an ordering on self adjoint operators, ordering on projection operators (Proposition 4.38).

*Revised: 5/21/2017*