Proof of Theorem 4.22

Note. In Promislow’s proof of Theorem 4.22, it is established that (see page 87)

\[ \psi_z(x) = \langle x, z \rangle = \langle x - \frac{f(x)}{f(z_0)} z_0, z \rangle + \frac{f(x)}{f(z_0)} \langle z_0, z \rangle \]

where \( z = f(z_0)z_0 \). It is then stated that: “The second term on the far right is equal to \( f(x) \).” However, this term in fact reduces to

\[ \frac{f(x)}{f(z_0)} \langle z_0, z \rangle = \frac{f(x)}{f(z_0)} \langle z_0, f(z_0)z_0 \rangle = \frac{f(z_0)}{f(z_0)} f(x). \]

Of course if the scalar field is \( \mathbb{R} \), then this is fine. However, we are not making this assumption.


Lemma. Let \( H \) be a Hilbert space and \( f \in H^* \) where \( f \neq 0 \). Then the dimension of \( N(f)^\perp \) is 1.

Proof. Since \( f \neq 0 \), then \( N(f) \neq H \). Since \( f \) is bounded, then \( f \) is continuous (Theorem 2.6), \( N(f) = f^{-1}(\{0\}) \), and so \( N(f) \) is closed (and “clearly” a subspace). So \( N(f) \) is a closed proper subspace of \( H \). Hence \( N(f)^\perp \) contains nonzero elements of \( H \). Let \( h_1, h_2 \in N(f)^\perp \). Since \( f(h_1) \neq 0 \) and \( f(h_2) \neq 0 \), then there exists some

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scalar $a \neq 0$ such that

$$f(h_1) + af(h_2) = f(h_1 + ah_2) = 0$$

(namely, $a = -f(h_1)/f(h_2)$). But then $h_1 + ah_2 \in N(f)$. But $N(f)\perp$ is a linear space, so $h_1 + ah_2 \in N(f)\perp$. So, by Theorem 4.14a, $h_1 + ah_2 = 0$. But then $h_1$ and $h_2$ are linearly dependent. So any two elements of $N(f)\perp$ are linearly dependent and the dimension of $N(f)\perp$ is 1.

\[\square\]

**Theorem 4.22.** For any $z$ in a Hilbert space $H$, the functional $\psi_z$ defined by $\psi_z(x) = \langle x, z \rangle$ is in $H^* = \mathcal{B}(H, \mathbb{F})$. Conversely, given any $f \in H^*$, there is a unique $z \in H$ such that $f = \psi_z$.

**Proof.** Since $\langle \cdot, \cdot \rangle$ is linear in the first position (by definition), then $\psi_z$ is linear. By the Cauchy-Schwartz Inequality (Theorem 4.3) $\|\psi_z(x)\| = \|\langle x, z \rangle\| \leq \|x\|\|z\|$, so $\|\psi_z\| \leq \|z\|$ (taking a supremum over all unit vectors $x$). With $z \neq 0$ and $x = z/\|z\|$ we have

$$\psi_z(x) = \langle z/\|z\|, z \rangle = \|z\|^2/\|z\| = \|z\|.$$  

So $\|\psi_z\| = \|z\|$. Therefore $\psi_z$ is bounded and so $\psi_z \in H^*$ for all $z \in H$.

By Lemma, there is a unit vector $z_0 \in N(f)\perp$. In fact, $N(f)\perp = \text{span}\{z_0\}$. Then for all $x \in H$, we have

$$x = x - \langle x, z_0 \rangle z_0 + \langle x, z_0 \rangle z_0. \quad (*)$$

Now since $N(f)\perp$ is one dimensional and $z_0$ is a unit vector, then $P_{N(f)\perp}(x) = \langle x, z_0 \rangle z_0$. By Theorem 4.14(b), for any $x \in H$ we have

$$x = P_{N(f)}(x) + P_{N(f)\perp}(x) = P_{N(f)}(x) + \langle x, z_0 \rangle z_0.$$
Therefore, $x - \langle x, z_0 \rangle z_0 = P_{N(f)}$ and $x - \langle x, z_0 \rangle z_0 \in N(f)$, or $f(x - \langle x, z_0 \rangle z_0) = 0$.

So from (⋆) we have

$$f(x) = f(x - \langle x, z_0 \rangle z_0 + \langle x, z_0 \rangle z_0)$$

$$= f(x - \langle x, z_0 \rangle z_0) + f(\langle x, z_0 \rangle z_0)$$

$$= 0 + \langle x, z_0 \rangle f(z_0)$$

$$= \langle x, f(z_0)z_0 \rangle \text{ (by conjugate linearity)}. $$

So with $z = f(z_0)z_0$, we have $f(x) = \langle x, z \rangle = \psi_z(x)$ for all $x \in H$. That is, $f = \psi_z$.

For uniqueness, suppose $f = \psi_z = \psi_{z'}$. Then for all $x \in H$, $\langle x, z \rangle = \langle x, z' \rangle$, or

$$\langle x, z \rangle - \langle x, z' \rangle = \langle x, z - z' \rangle = 0.$$ 

So $z - z' \in H^\perp = \{0\}$, and $z = z'$.

\[ \blacksquare \]

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