Proof of Theorem 4.22

Note. In Promislow's proof of Theorem 4.22, it is established that (see page 87)

$$\psi_z(x) = \langle x, z\rangle = \left\langle x - \frac{f(x)}{f(z_0)} z_0, z \right\rangle + \frac{f(x)}{f(z_0)}\langle z_0, z\rangle$$

where $z = f(z_0) z_0$. It is then stated that: "The second term on the far right is equal to $f(x)$." However, this term in fact reduces to

$$\frac{f(x)}{f(z_0)}\langle z_0, z\rangle = \frac{f(x)}{f(z_0)}\langle z_0, f(z_0)z_0\rangle = \frac{\left(\overline{f(z_0)}\right)}{f(z_0)} f(x).$$

Of course if the scalar field is $\mathbb{R}$, then this is fine. However, we are not making this assumption.

Note. This handout gives a proof of Theorem 4.22 based in part on the argument given in Lokenath Debnath and Piotr Mikusiński's *Introduction to Hilbert Spaces with Applications*, 3rd Edition, Elsevier Press (2005). See pages 133 and 134.

Lemma. Let H be a Hilbert space and $f \in H^*$ where $f \neq 0$. Then the dimension of $N(f)^\perp$ is 1.

Proof. Since $f \neq 0$, then $N(f) \neq H$. Since f is bounded, then f is continuous (Theorem 2.6), $N(f) = f^{-1}(\{0\})$, and so $N(f)$ is closed (and "clearly" a subspace). So $N(f)$ is a closed proper subspace of H. Hence $N(f)^\perp$ contains nonzero elements of H. Let $h_1, h_2 \in N(f)^\perp$. Since $f(h_1) \neq 0$ and $f(h_2) \neq 0$, then there exists some scalar $a \neq 0$ such that

$$f(h_1) + af(h_2) = f(h_1 + ah_2) = 0$$

(namely, $a = -f(h_1)/f(h_2)$). But then $h_1 + ah_2 \in N(f)$. But $N(f)^\perp$ is a linear space, so $h_1 + ah_2 \in N(f)^\perp$. So, by Theorem 4.14a, $h_1 + ah_2 = 0$. But then h_1 and h_2 are linearly dependent. So any two elements of $N(f)^\perp$ are linearly dependent and the dimension of $N(f)^\perp$ is 1. ∎

Theorem 4.22. For any z in a Hilbert space H, the functional ψ_z defined by $\psi_z(x) = \langle x, z\rangle$ is in $H^* = \mathcal{B}(H, \mathbb{F})$. Conversely, given any $f \in H^*$, there is a unique $z \in H$ such that $f = \psi_z$.

Proof. Since $\langle \cdot, \cdot \rangle$ is linear in the first position (by definition), then ψ_z is linear. By the Cauchy-Schwartz Inequality (Theorem 4.3) $\|\psi_z(x)\| = \|\langle x, z\rangle\| \leq \|x\|\|z\|$, so $\|\psi_z\| \leq \|z\|$ (taking a supremum over all unit vectors x). With $z \neq 0$ and $x = z/\|z\|$ we have

$$\psi_z(x) = \langle z/\|z\|, z\rangle = \|z\|^2/\|z\| = \|z\|.$$

So $\|\psi_z\| = \|z\|$. Therefore ψ_z is bounded and so $\psi_z \in H^*$ for all $z \in H$.

By Lemma, there is a unit vector $z_0 \in N(f)^\perp$. In fact, $N(f)^\perp = \text{span}\{z_0\}$. Then for all $x \in H$, we have

$$x = x - \langle x, z_0\rangle z_0 + \langle x, z_0\rangle z_0. \qquad (*)$$

Now since $N(f)^\perp$ is one dimensional and z_0 is a unit vector, then $P_{N(f)^\perp}(x) = \langle x, z_0\rangle z_0$. By Theorem 4.14(b), for any $x \in H$ we have

$$x = P_{N(f)}(x) + P_{N(f)^\perp}(x) = P_{N(f)}(x) + \langle x, z_0\rangle z_0.$$

Therefore, $x - \langle x, z_0\rangle z_0 = P_{N(f)}$ and $x - \langle x, z_0\rangle z_0 \in N(f)$, or $f(x - \langle x, z_0\rangle z_0) = 0$. So from $(*)$ we have

$$\begin{aligned} f(x) &= f(x - \langle x, z_0\rangle z_0 + \langle x, z_0\rangle z_0) \\ &= f(x - \langle x, z_0\rangle z_0) + f(\langle x, z_0\rangle z_0) \\ &= 0 + \langle x, z_0\rangle f(z_0) \\ &= \langle x, \overline{f(z_0)} z_0\rangle \text{ (by conjugate linearity).} \end{aligned}$$

So with $z = \overline{f(z_0)} z_0$, we have $f(x) = \langle x, z\rangle = \psi_z(x)$ for all $x \in H$. That is, $f = \psi_z$.

For uniqueness, suppose $f = \psi_z = \psi_{z'}$. Then for all $x \in H$, $\langle x, z\rangle = \langle x, z'\rangle$, or

$$\langle x, z\rangle - \langle x, z'\rangle = \langle x, z - z'\rangle = 0.$$

So $z - z' \in H^{\perp} = \{0\}$, and $z = z'$. ∎

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