The Bicentennial of Evariste Galois

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[**References**]

[**"The Question" slide**]

[**The Equation**] In modern notation, this presentation is about expressing the solutions of the equation:

$$c_n x^n + c_{n-1}x^{n-1} + c_{n-2}x^{n-2} + \cdots c_3 x^3 + c_2 x^2 + c_1 x + c_0 = 0$$

in an algebraic way. [**INSERT Algebraic**] That is, what does x equal? Express the answer as a combination of the coefficients using the operations of addition, subtraction, multiplication, division, and extraction of roots. [**Insert Radicals**] An algebraic solution is also sometimes called "solving by radicals."

[**Familiar Example**] A familiar example of the problem is to find the roots (or zeros) of the second degree equation $ax^2+bx+c=0$. We want the values of x expressed in terms of a, b, and c and only with the use of addition, subtraction, multiplication, division, and the extraction of roots. Of course, the values of x are given by the familiar quadratic equation: $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$.

[**Unfamiliar Example.** Another example, which is likely not so familiar to you, is that of finding the values of x which result from setting a third degree polynomial equal to zero: $ax^3+bx^2+cx+d=0$. Again, the values of x must be expressed in terms of the coefficients a, b, c, and d by using the algebraic operations. The reason you probably have not seen this in the past is that it is actually rather impractical and complicated.

[**Roots of a Cubic**] Here are the solutions. Notice that x_2 and x_3 involve square roots of negatives.

[**Fourth Degree and Fifth Degree**] There is a similar algebraic equation for the four solutions of the equation which results from setting a fourth degree polynomial equal to zero. However, there is no such algebraic equation for the solutions of a fifth degree polynomial set equal to zero. This talk is a historical exploration of a key figure related to this second result.

[**Brief History of Equations: Quadratics**]

[**Map of Babylon**] The Babylonians did not use algebraic symbols in the modern sense. Instead, they would state problems entirely in terms of words. The solution would be verbal instructions explaining how to solve the given problem, but there was no sense of generality or a formula. Many of the problems concerned partitioning of land and so involved areas and quadratics.

[**Map of Egypt**] The ancient Egyptians could only solve quadratic equations involving x^2 terms and constant terms, but not "mixed" equations involving both x^2 and x terms.

[**Diophantus**] Diophantus is sometimes called the 'father of algebra.' Little is known about Diophantus—he lived sometime between 150 and 250 CE. His major work was *Arithmetica*, which mostly deals with number theory. His main contribution is to move forward the symbolic form of equations from the purely rhetorical style of the Babylonians to something more similar to our modern notation.

[**Brahmagupta**] The Indian mathematician and astronomer Brahmagupta was the first to solve quadratic equations that involved negative numbers. He stated the rules for multiplying or dividing positive and negative numbers as: [**Insert quote**] "The product or ratio of two debts is a fortune; the product or ratio of a debt and a fortune is a debt."

[**al-Khwarizmi**] Muhammad ibn Musa al-Khwarizmi literally gave algebra its name when he published in Baghdad: [**Insert book title**] *Kitab al-jabr we al-muqabalah* (The Condensed Book on Restoration and Balancing). In this book, the first systematic solution of quadratic equations was given, and this book remained the quintessential reference on the theory of equations for centuries.

[**Ha-nasi**] The quadratic equation was finally published in its general form in Europe in the 12th century by Abraham bar Hiyya Ha-nasi in his book *Treatise on Measurement and Calculation*.

[**Brief History of Equations: Cubic**]

[**dal Ferro/Fiore**] In 1515, Scipione dal Ferro managed to solve cubic equations of the form $ax^3 + bx = c$. He did not publish his results, but told them to his student Antonio Maria Fiore. At this point in time, one could attain prestige and promotions by participating in public mathematical contests. In 1535, Fiore challanged Niccolò Tartaglia to a "cubic solving contest."

[**Tartaglia**] Each contestant posed 30 problems for the other contestant to solve. All of Fiore's problems were of the form $ax^3 + bx = c$. Tartaglia's problems were of a much greater variety. Tartaglia solved all of Fiore's problems in about 2 hours, and Fiore could not solve *any* of Tartagli's problems. This contest took Tartaglia from obscurity to the level of a mathematical celebrity. It is at the time of this contest that Tartaglia found the general solution to the cubic equation.

[**Brief History of Equations: Quartic**]

[**Ferrari**] In 1540, Lodovico Ferrari solved the quartic equation.

[**Cardano**] Five years later, in 1545 Gerolamo Cardano published the book considered as the beginning of algebra as it is done today. The book was *Ars magna* (*The Great Art*). The cubic and quartic equations are explored in detail. Cardano and Tartaglia were involved in one of the most contentious episodes in the history of math, arguing over the publication of solutions to some of the cubic equations. None-the-less, the publication of *Ars Magna* and the rapid succession of the solutions of the cubics and quartics lead many to think that solutions of higher order equations were imminent.

[**Brief History of Equations: Quintic**]

[**Unsuccessful**] A number of people unsuccessfully attempted to solve the quintic: Scot James Gregory, Ehrenfried Walther von Tschirnhaus, Etienne

Bezout, and Erland Samuel Bring.

[**Rufinni**] Paolo Ruffini published a two-volume, 516 page book, *General Theory of Equations*, in which he claimed to prove that there was no algebraic formula for the solutions to the quintic equation. However, the lengthy argument was difficult to follow and Ruffini tried to clarify his proof in new publications in 1803 and 1806. His proof never became widely known nor accepted. Subsequent study of the work reveals that he did not quite prove the claim.

[**Abel**] In 1823, Norwegian mathematician Niels Henrik Abel ended the discussion of the question of our interest. He successfully *proved* that there is no algebraic equation to give the solutions to a general fifth degree polynomial equation. He was not aware of Ruffini's work. Abel was very poor. He published the result himself as a pamphlet in French and reduced his argument to only six pages. The pamphlet found no audience. In 1827, Abel published two papers on elliptic functions and introduced a technique to study the use of these non-algebraic functions to find solutions to polynomial equations. Abel's life is quite tragic and he died of tuberculosis in 1829 at the age of 26.

[**The Real Question**] The question that Abel did *not* solve is the one of our interest: "When is there an algebraic equation which gives the solutions of a polynomial equation?" That is, what are the conditions on the coefficients of a polynomial which guarantee that the solutions to the resulting polynomial equation can be expressed in terms of the coefficients using addition/subtraction, multiplication/division, and extraction of roots? The solution to this question was given by an even more tragic figure than Abel... and even younger at the time of his success!

[**Galois**] Two hundred years ago on October 25, 1811, Evariste Galois was born to Nicolas-Gabriel Galois and Adélaide-Marie Galois.

[**Paris Map**] The family lived in Bourg-la-Reine, a few miles south of Paris, where the father, Nicalos-Gabriel, was mayor and a well-known leader of the liberal movement.

[**France of Galois**] This was a time of political unrest in France. In 1814, Emperor Napoleon Bonaparte abdicated and Louis XVIII stepped in as king promising a constitutional monarchy. However, he concentrated power, strengthened state connections with the Catholic Church, allowed the rich two votes in elections, and censored the press. This lead to the formation of an opposition of those who saw themselves as supporters of the Revolution of 1789. Young Evariste Galois would become involved in the liberal movement.

[**Charles X**] In 1824, Louis XVIII died and was replace by Charles X. He abolished press censorship, but strengthened the connection with the Catholic Church and introduced crime against religion into the penal code and made sacrilege a capital offense.

[**Timeline**] In 1823, Evariste left home and was admitted to the famous Parisian boarding school the *Lycée Louis-le-Grand*.

[**Lycee Louis-le-Grand**] Galois was awarded a prize and other distinctions during his first year at the Lycee. During his second year, he received recognition in Greek and won first prize in Latin poetry. In his third year he suffered from a serious earache that lasted throughout the winter, likely caused by the cold damp conditions in which he lived.

[**Legendre**] Adrien-Marie Legendre's *Elements of Geometry* made him internationally famous. Galois *discovered* mathematics when he read this book and one can rightly say that it changed his life. In fact, there is a legend that Galois read the book, which was intended for a two-year sequence, in only two days. After reading the book, Galois lost interest in all subjects, except mathematics. He soon realized that he was different from his schoolmates, read more, and started to work on original research. In fact, he thought he had found an algebraic solution to the quintic equation, only to later find an error in his computations.

[**Richard**] During the 1828-29 academic year, Galois took a special mathematics class from Louis-Paul-Emile Richard. Richard recognized Galois' talent. Galois regularly did his homework and supplied original solutions which Richard kept and used when teaching subsequent classes. In fact, these solutions survive to today and are now in the Institute of France. This time with Richard probably represents some of the happiest of Galois' life.

[**Timeline: First Paper**] Richard helped Galois publish his first paper, "Proof of a Theorem on Periodic Continued Fractions"appeared in the April 1, 1829 issue of *Annals of Pure and Applied Mathematics*.

[**Two Questions**] Galois, not knowing of the work of Abel, started working on two of the same problems which Abel explored: (1) Find all the equations of a specific degree that are solvable by algebra (i.e., by radicals), and (2) Decide whether a given equation can be solved by algebra or not. [**Insert: Memoires**] The results of Galois' work were two memoires which Richard proposed could best be judged by the French Academy of Sciences.

[**The Academy 1**] Augustine Cauchy became a fellow of the French Academy of Sciences in 1816, and had only presented his own research to the Academy since then. However, on May 25, 1829 he presented Galois' memoire "Algebraic Research" to the Academy. It was to be judged by Cauchy himself, along with [**Insert Fourier Navier**] Joseph Fourier and Claude Navier.

[**The Academy 2**] On June 1, 1829 Cauchy presented Galois' second memoire, "Research on Algebraic Equations of the First Degree." Judgment was given to Cauchy and Poisson. At this point, Cauchy was an extremely well-respected mathematician, and the Academy allowed him to take the two memoires home.

[**Timeline: Father's Death**] The political climate in Paris at this time was still heated, and a plot was hatched to force Evariste's father out of the office of mayor in Bourg-la-Riene. Rumors were spread and documents forged, and Nicalos-Gabriel Galois was forced out as mayor and he moved to Paris. On July 2, 1829, in a state of depression, he committed suicide. At his funeral, there were protests and rock throwing. Evariste was there to witness this.

[**Ecole Polytechnique**] Following his time at the Lycee Louis-le-Grand, Galois aspired to gain admission to the Ecole Polytechnique, the lead research university in France.

[**Timeline: Fail Exam**] Unfortunately, Galois failed the entrance exam when he first took it in June 1828. He took the exam a second time in July 1829, shortly after his father's death. [**Insert: Fail 2**] Galois became angered that the testers did not understand his answers and threw a blackboard eraser at one of them. He failed again. The entrance exam could only be taken twice, so he would not be attending the Ecole Polytechnique.

[**Timeline: Ecole Preparatoire**] If he was not a student, then Galois would have no way to support himself. On February 20, 1830, he signed a pledge to become a teacher and remain in the education system for 10 years. This allowed him to enter the Ecole Preparatorie. It was at this school that he met his good friend Auguste Chevalier, who would have an influence on his political views and his mathematical legacy.

[**Cauchy '?'**] During this time, Galois had heard nothing concerning his memoires which he had submitted to the Academy. In fact, Cauchy never again mentioned Galois. Apparently Cauchy had lost the memoires!

[**Grand Prix**] In mid-1830 the French Academy announced a math contest (the *Grand Prix de Mathématiques*). Galois rewrote his paper and presented his paper to the Academy. However, the "Galois luck" kicked in again! Joseph Fourier took that paper home and, a few weeks later died. [**Insert X's**] Galois' paper was not considered, but no one bothered to tell this to Galois.

[**Poisson**] With the encouragement of Poisson, Galois submitted his memoire of algebraic equations a third time. On July 4, 1831 the paper was given a negative ruling. The opinion was that the work was wrong. However, a careful review of the report reveal errors in the ruling. They wrote: "We made all possible efforts to understand M. Galois' evidence. His thesis is neither clear enough, nor sufficiently developed to enable us to judge its rigour. Neither are we able to provide a clear idea of this work. For this reason, we return your manuscript in the hope that you will find M. Poisson's observations of use for your future research."

[**Galois' Papers**] Mid-1830 did see a bit of recognition for Galois' work. [**Insert 2nd**] In April 1830, he published "An Analysis of the Resolution of Algebraic Equations" in *Bulletin of the Mathematical, Physical, and Chemical Sciences* **13**, 271–272. Two of his papers appeared in the June 1830 issue of the same journal: [**Insert 3rd**] "Note on the Resolution of Numerical Equations" in **13**, 413–414; and [**Insert 4th**] "On the Theory of Numbers" in **13**, 428–435. In fact, Galois published five papers during his life. [**Insert 1st**] The first was mentioned above, and the last one is: [**Insert 5th**] "Notes on Some Points of Analysis" in *Annals of Pure and Applied Mathematics* **11**, 182–184 (December 1830). Some biographers (such as E.T. Bell in *The Men of Mathematics*) have portrayed Galois as one entirely outside of the mathematical community, and this is not the case, as evidenced by his research record.

[**More Politics**]

[**Charles X**] In reaction to a parliamentary vote in support of the constitution, Charles X reacted by dissolving Parliament in July 1830. He suspended freedom of press, annulled recent elections, and introduced a property qualification for voting. This lead to a revolt in the streets of Paris, with 2000 dead in the king's army and 1800 dead citizens. As a result, Charles X was replaced by Louis-Phillipe I, "King of the French." Galois, was distraught that he could not join the protests because he was locked in at the Ecole Preparatorie.

[**Letter to a Newspaper**] Angered by what he perceived as political oppression at the Ecole Preparatorie, Galois wrote a letter to a Paris newspaper insulting one of the officials at the school. This lead to his expulsion in December 1830. This freed time up for Galois to concentrate full-time on politics. He joined the rebellious National Guard and was issued a uniform.

[**Society of Friends**] He joined the "Society of Friends of the People" which was recently established as an activist organization for those most interested in the republican ideals. The Society started to arm itself under the cover of the artillery of the National Guard.

[**Louis-Phillipe**] In December 1830, Louis-Phillipe dismissed the popular General Lafayette and disbanded the National Guard.

[**Trial of the Nineteen**] Many of the National Guard refused to hand over their arms, and nineteen of the artilleryman who were seen as leaders, were arrested. They went to trial in April 1831 and were found not guilty.

[**The Banquet**] To celebrate the verdict, the Society organized a banquet for May 9, 1831. It was attended by 200 of those most hostile to the government. At one point, Galois stood to make a toast. He raised a glass of wine in one hand and a jackknife in the other and toasted "To Louis-Phillipe!" Several of the guests raised their glasses and repeated the threatening toast. However, many were concerned with the tone of the gesture and quickly left. The banquet ended shortly afterwards in confusion.

[**Galois in Court**] Galois was arrested the next day, charged with incitement to an attempt on the life and person of the King of the French. He was tried on June 15, 1831. His defense was: "This is what happened: I had a knife which I had used to cut meat during the meal. I brandished it saying 'To Louis-Phillipe, if he betrays us!' These words were only heard by the people next to me, considering the whistling that had begun after the beginning of my utterance, because people thought I was proposing a toast to the good health of Louis-Phillipe." He was found not guilty.

[**Re-Arrest**] The republicans were organizing demonstrations for the celebration of Bastille day on July 14, 1831. Arrests were made of those thought to be the most threatening. Galois was found wearing his old National Guard uniform and carrying several weapons. He was arrested and charged with possession of illegal arms and wearing a uniform to which he was not entitled. He was detained for three months until his trial in October 1831. He was found guilty, appealed, and lost. He would remain in jail until April 1832.

[**In Prison**] Galois spent most of his time in Sainte-Pélagie Prison. He passed much of the time by walking up and down the yard, in meditation. He sometimes drank to excess with his cell-mates, and would then faint. His sister Nathalie-Thédore visited him often in the prison.

[**Stephanie**] In the spring of 1832, a cholera epidemic swept through Paris. On March 16 Galois was transferred as a prisoner on parole to a nearby clinic. A doctor by the name of Jean-Pouis Poterin-Dumotal, who lived with his family in the same street, worked there. Galois met the doctor's daughter, Stéphanie, and fell in love with her. The only evidence of the relationship are two letters from Stéphanie, that Galois must have torn up in a moment of rage. He regretted this later, and tried to reconstruct the content. The letters indicate that initially Stéphanie had encouraged Galois, but later lost interest. Galois was released from prison on April 29, 1832. He had one month left to live.

[**Letters**] On the evening of May 29, 1832, Evariste Galois composed three letters. He was to fight a pistol duel the next morning and his letters speak of his impending death. [[**Insert 1st Letter**] The first letter is addressed *to all republicans* and states: "I beg my patriotic friends not to chide me for dying in any other way than for my country. I die, the victim of a cruel coquette, and of two of her victims. My life fades away amidst trivial gossip. ... Adieu! Life was dear to me, for the common good. Pardon for those who killed me. They were acting in good faith." [[**Insert 2nd Letter**] The second letter is addressed only to the initials "N.L." It states: "I have been challenged to a duel by two patriots ... I cannot refuse." [[**Insert 3rd Letter**] The third letter is to his friend, Auguste Chevalier, and concerns mathematics. He gave a brief summary of the *mémoire* he had deposited at the Academy, adding some new theorems and conjectures covering seven pages. He concluded: "**I do not have enough time** and my ideas are not sufficiently well developed in this area, which is enormous."

[**The Duel**] On the morning of May 30, 1832 at a distance of 25 paces, Evariste Galois was shot in the abdomen. He did not die instantly. He was taken to the Cochin Hospital by someone. It is unclear who took him to the hospital; some think it was a peasant on the way to market or a former officer of the royal army.

[**May 31, 1832**] Evariste's brother Alfred rushed to his brother's side when he heard of the shooting. Alfred was there to hear Evariste's last words: "Don't cry. I need all my courage to die at twenty." He died of peritonitis at 10:00 in the morning on Thursday May 31, 1832. His death was announced in the newspapers of Paris the following day. The record of the events consists of these newspaper articles and the coroner's report.

[**Myths**] Many myths about Galois exist. One is that his work consists entirely of writing down his theories on the night before he died. As we've seen, he published during his life and this is not exactly the case. Some of this is based on the fact that there have been dramatizations of Galois' life, both books and movies. In addition, the details of the duel are very unclear. Who fired the fatal shot?

[**Rigatelli**] A well-researched biography due to Laura Rigatelli was published in Italian in 1996. Her theory is as follows. Galois was passionate about the republican movement. At one point, he tells his family if only a body would cause an uprising that would lead to the establishment of a republic and the end of the monarchy, he would gladly offer his life. Rigatelli proposes that the duel was a shame with the purpose of producing the body of a well-known revolutionary around which the republicans could circle. At the funeral, those in attendance would start an uprising. However, during the funeral, it became known that General Lafayette had died and the decision was collectively made to wait to riot until Lafayette's funeral, which would have a larger attendance. Galois' death had been in vain.

[**Livio**] Mario Livio does not find Rigatelli's argument compelling, and takes a more traditional view, particularly focusing on the "infamous coquette" mentioned in Galois' first letter. He proposes that Galois' love interest, Stephanie, had been offended and that the duel was arranged by two of her uncles. This is in line with the main-stream idea that Galois "died in a duel over a woman." Livio's contribution is to speculate on particular names who were responsible for Galois' death.

[**Galois Theory**]

[**Three Geometry Problems**] Galois' ideas *did* eventually catch on during the 19th century. Not only does his ideas apply to algebraic solutions of equations, but also to geometric constructions. Galois theory can be used to show that the three classic construction problems from geometry *cannot* be performed:

Squaring the circle: Drawing a square the same area as a given circle.

Doubling the cube: Drawing a cube with twice the volume of a given cube.

Trisecting the angle: Dividing a given angle into three smaller angles all of the same size.

[**Modern Galois Theory**] In modern terms, Galois' main result can be stated as: "The polynomial equation $p(x) = 0$ is solvable by radicals if and only if the Galois group of p is solvable." This requires constructing chains of normal subgroups of the Galois group associated with the equation. In a real sense, Galois is the one (along with Abel) who gave birth to the modern algebra we study as undergraduates and graduates—that is, the areas of groups, fields, and extension fields.

[**Today**] Galois Theory is still an active area of research. The American Mathematical Society Classification Number for Galois Theory is 11R32. Conferences are still held on Galois theory. An upcoming conference of interest will be held in a couple of weeks at the Henri Poincaré Institute in Paris. It seems to be the only conference scheduled specifically with the Galois bicentennial in mind.

[**Conclusion**] Evariste Galois is remembered 200 years later having had a huge impact on the advancement of both mathematics and physics (which has lately turned its attention to the study of symmetry and therefore of group theory). His contributions were complete before he was 21 years old–a humbling realization for must of us since we are at least that age; some of us 2 or 3 times over! So we wish him a happy birthday!

[**References**]