Chapter 3. Conic Sections
3.1. The Parabola—Proofs of Theorems
1 Theorem 3.1 (Apollonius’ Proposition I.11)
Theorem 3.1 (Apollonius’ Proposition I.11)

**Theorem 3.1.** (Apollonius’ Proposition I.11 in *Treatise on Conic Sections*)
If a cone is cut by a plane that has the same slope as the generators of the cone, then the intersection is a parabola.

**Proof.** For a point $P$ outside of a sphere, all line segments from $P$ to the sphere which are tangent to the sphere are of the same length (we take this as true, and refer to Figure 3.2 left). These segments form a cone and intersect the sphere in a circle.
Theorem 3.1 (Apollonius’ Proposition I.11)

**Theorem 3.1.** (Apollonius’ Proposition I.11 in *Treatise on Conic Sections*) If a cone is cut by a plane that has the same slope as the generators of the cone, then the intersection is a parabola.

**Proof.** For a point $P$ outside of a sphere, all line segments from $P$ to the sphere which are tangent to the sphere are of the same length (we take this as true, and refer to Figure 3.2 left). These segments form a cone and intersect the sphere in a circle.

![Diagram showing a parabola as the intersection of a cone with a plane](image)
Theorem 3.1 (Apollonius’ Proposition I.11)

Theorem 3.1. (Apollonius’ Proposition I.11 in Treatise on Conic Sections)
If a cone is cut by a plane that has the same slope as the generators of the cone, then the intersection is a parabola.

Proof. For a point $P$ outside of a sphere, all line segments from $P$ to the sphere which are tangent to the sphere are of the same length (we take this as true, and refer to Figure 3.2 left). These segments form a cone and intersect the sphere in a circle.
Theorem 3.1 (Apollonius’ Proposition I.11); Continued 1

Proof (continued). Next, introduce a Dandelin sphere in the cone which touches the cone in a circle $C$, as shown in Figure 3.2 (center). Then introduce a plane $\pi$ that is parallel to a generator (that is, $\pi$ is parallel to some ray which starts at vertex $P$ and lies on the cone) and is tangent to the sphere at point $F$ (alternatively, we could introduce plane $\pi$ first and then the Dandelin sphere). Let $P$ be an arbitrary point on the intersection of the cone with the plane. Let point $A$ be the point on the intersection of the circle $C$ with the generator of the cone that contains point $P$. 
Theorem 3.1 (Apollonius’ Proposition I.11); Continued 1

Proof (continued). Next, introduce a Dandelin sphere in the cone which touches the cone in a circle $C$, as shown in Figure 3.2 (center). Then introduce a plane $\pi$ that is parallel to a generator (that is, $\pi$ is parallel to some ray which starts at vertex $P$ and lies on the cone) and is tangent to the sphere at point $F$ (alternatively, we could introduce plane $\pi$ first and then the Dandelin sphere). Let $P$ be an arbitrary point on the intersection of the cone with the plane. Let point $A$ be the point on the intersection of the circle $C$ with the generator of the cone that contains point $P$.

![Diagram of a parabola as the intersection of a cone with a plane](image-url)
Proof (continued). Next, introduce a Dandelin sphere in the cone which touches the cone in a circle $C$, as shown in Figure 3.2 (center). Then introduce a plane $\pi$ that is parallel to a generator (that is, $\pi$ is parallel to some ray which starts at vertex $P$ and lies on the cone) and is tangent to the sphere at point $F$ (alternatively, we could introduce plane $\pi$ first and then the Dandelin sphere). Let $P$ be an arbitrary point on the intersection of the cone with the plane. Let point $A$ be the point on the intersection of the circle $C$ with the generator of the cone that contains point $P$.

Fig. 3.2. A parabola as the intersection of a cone with a plane
Proof (continued). Now the plane containing the circle $C$ intersects plane $\pi$ in a line $\ell$; we will see that this line is the directrix of the parabola. Let $B$ be the point on the directrix above point $P$ (so that segment $BP$ is perpendicular to line $\ell$). See Figure 3.2 (right). Then segment $PA$ is tangent to the sphere (since it is part of a generator) and $PF$ is tangent to the sphere since it lies in plane $\pi$ and $\pi$ is tangent to the sphere at point $F$ (by construction). Hence, as observed above, this means that segments $PA$ and $PF$ are of the same length.
Proof (continued). Now the plane containing the circle $C$ intersects plane $\pi$ in a line $\ell$; we will see that this line is the directrix of the parabola. Let $B$ be the point on the directrix above point $P$ (so that segment $BP$ is perpendicular to line $\ell$). See Figure 3.2 (right). Then segment $PA$ is tangent to the sphere (since it is part of a generator) and $PF$ is tangent to the sphere since it lies in plane $\pi$ and $\pi$ is tangent to the sphere at point $F$ (by construction). Hence, as observed above, this means that segments $PA$ and $PF$ are of the same length.
Proof (continued). Now the plane containing the circle $C$ intersects plane $\pi$ in a line $\ell$; we will see that this line is the directrix of the parabola. Let $B$ be the point on the directrix above point $P$ (so that segment $BP$ is perpendicular to line $\ell$). See Figure 3.2 (right). Then segment $PA$ is tangent to the sphere (since it is part of a generator) and $PF$ is tangent to the sphere since it lies in plane $\pi$ and $\pi$ is tangent to the sphere at point $F$ (by construction). Hence, as observed above, this means that segments $PA$ and $PF$ are of the same length.

Fig. 3.2. A parabola as the intersection of a cone with a plane
Proof (continued). The slope of the line segment $PF$ as given in Figure 3.2 (center) is the same as the slope of line segment $PA$ when viewed in a vertical plane containing the generator containing $PA$ (since plane $\pi$ has the same “slope” as the a generator of the cone, as Ostermann and Wanner put it; see page 63). Since the vertical displacement form $P$ to $A$ is the same as the vertical displacement from $P$ to $B$, then by congruent triangles the length of $PA$ equals the length of $PB$. Therefore the length of $PF$ equals the length of $PB$ and hence the distance from point $P$ to the focus $F$ is the same as the distance of $P$ to the directrix.
Proof (continued). The slope of the line segment $PF$ as given in Figure 3.2 (center) is the same as the slope of line segment $PA$ when viewed in a vertical plane containing the generator containing $PA$ (since plane $\pi$ has the same “slope” as the a generator of the cone, as Ostermann and Wanner put it; see page 63). Since the vertical displacement form $P$ to $A$ is the same as the vertical displacement from $P$ to $B$, then by congruent triangles the length of $PA$ equals the length of $PB$. Therefore the length of $PF$ equals the length of $PB$ and hence the distance from point $P$ to the focus $F$ is the same as the distance of $P$ to the directrix.
Proof (continued). The slope of the line segment $PF$ as given in Figure 3.2 (center) is the same as the slope of line segment $PA$ when viewed in a vertical plane containing the generator containing $PA$ (since plane $\pi$ has the same “slope” as the a generator of the cone, as Ostermann and Wanner put it; see page 63). Since the vertical displacement form $P$ to $A$ is the same as the vertical displacement from $P$ to $B$, then by congruent triangles the length of $PA$ equals the length of $PB$. Therefore the length of $PF$ equals the length of $PB$ and hence the distance from point $P$ to the focus $F$ is the same as the distance of $P$ to the directrix.