Theorem 5.8 (Newton’s Theorem 1)

“The areas, which revolving bodies describe by radii drawn to an immovable centre of force, do lie in some immovable planes, and are proportional to the times in which they are described.”

Proof. We go through the argument given in Ostermann and Wanner (which is the same as given by Newton in *Principia*). The argument is a discretized version of the physical problem. Newton appeals to another result (Lemma III in his Book I) to move from the discrete to the continuous case. We place the Sun at point $S$ (the “centre of force”) and approximate the orbit with points $A$, $B$, $C$, $D$, $E$, etc. The force $f$ then acts on the object orbiting the Sun by pulling the object in the direction of the Sun. We let the force act on the object at point $A$ and then assume the object moves in a straight line over time period $\Delta t$ without the form acting from $A$ to $B$ (the motion is along a straight line by Newton’s First Law, or “Lex 1”). See the figure below.

Proof (continued). Next, we let the force act again at point $B$ by an amount $f \Delta t$. Now consider the triangles $ABS$ and $BcS$ in the figure. We claim that these triangles have the same areas. We argue this below.

The Figure for Newton’s Theorem 1 from Motte’s 1846 translation of Principia.
Theorem 5.8 (Newton’s Theorem 1, continued 3)

**Proof (continued).** Next, we let the force affect the object at point $B$. By Newton’s Second Law (or Lex 2), the force acts along the line from $B$ to $S$; the “change of motion” (as it’s called in Lex 2) is represented as the vector from $B$ to $V$ (we denote this vector, and similar vectors, as $BV$) in Figure 5.29 (right) below. Adding this change in motion to the velocity of the object before the application of the force (represented by the vector $AB$ in the figure; think of it as the change in position over time $\Delta t$), we get the new velocity by adding vectors $AB$ and $BV$ to get the resultant vector $AV$. We translate $AV$ to point $B$ to get velocity vector $BC$.

![Diagram](image1)

Theorem 5.8 (Newton’s Theorem 1, continued 5)

**Proof (continued).** Since, as shown above, the triangles $ABS$ and $BCS$ have the same areas, then we have that triangles $ABS$ and $BCS$ have the same areas. Similar, the areas of triangles $ABS$, $CBS$, $CDS$, $DES$, etc. are all the same.

So the claim holds when we have the discrete version of the problem with the force applied as impulses at equal time steps $\Delta t$.

![Diagram](image2)

The Figure for Newton’s Theorem 1 from Motte’s 1846 translation of Principia.

Theorem 5.8 (Newton’s Theorem 1, continued 6)

**Theorem 5.8.** (Theorem 1 of Newton’s *Principia Mathematica.*) “The areas, which revolving bodies describe by radii drawn to an immovable centre of force, do lie in some immovable planes, and are proportional to the times in which they are described.”

**Proof (continued).** To complete the proof, Newton now applies his Lemma III which, in essence, involve taking a limit as $\Delta t \to 0$. Ostermann and Wanner describe Newton’s approach as similar to the numerical technique called the “Euler method” which is used to approximate solutions to differential equations.

In the spirit of Calculus 1 (MATH 1910), we could consider the area swept out by the line segment joining the Sun and the orbiting object as a function time $A(t)$. We then have from Newton’s discrete argument that $\Delta A/\Delta t$ is a constant. So we have $\lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} = \frac{dA}{dt}$ is constant (this is the constant of proportionality in Newton’s theorem). □
Newton’s Lemma. Let $APQ$ be an ellipse with focus $S$ and suppose $P$ to be the position of the planet moving towards $Q$, while the point $R$ moves on the tangent with $S$, $Q$, $P$ collinear. Let $T$ be the orthogonal projection of $Q$ onto $PS$ (see Figure 5.30, right). Then if the distance $PQ$ tends to zero, we have $RQ \approx (\text{Constant}) \cdot QT^2$, where the constant is independent of the position of $P$ on the ellipse.

Fig. 5.30. Reproductions from Newton’s autograph (1684), manuscript Cambridge Univ. Lib. Add. 3965; the force acting on a moving body (left); picture for Newton’s lemma (right). Reproduced by kind permission of the Syndics of Cambridge University Library

Proof.

From this figure, we have by Apollonius II.6 (see Section 3.2. The Ellipse, Figure 3.7(b)) that the tangent $PR$ is parallel to the “diameter” $DCK$ which is conjugate to diameter $GCP$. Let the lengths of these diameters be $2d$ and $2c$, respectively, as labeled in the Figure 5.31. Through focus $H$ we draw a parallel to $DK$ (light dotted line) to determine point of intersection $I$ with segment $SP$. Through point $Q$ we draw a parallel to $DK$ (light solid line) to determine point of intersection $V$ with segment $CP$ (see the inset in Figure 5.31).

By Apollonius III.48 (see Section 3.2. The Ellipse), the normal to $DK$ from point $F$ to point $P$ (let $PF = h$) is a bisector of angle $SPH$. That is, triangle $IPH$ is an isosceles triangle and so (by Euclid I.6) $IP = PH$. Since $SC = CH$, then by Thales’ Intercept Theorem (Theorem 1.1) we have $SE = EI$. By Apollonius III.52 (the alternative definition of an ellipse from Section 3.2. The Ellipse), we have $SE + EI + IP + PH = 2a$. Substituting $EI$ for $SP$ and substituting $IP$ for $PH$, we have: $EP = EI + IP = a$. (5.55)

Proof (continued).

If the ellipse were in fact a circle, then by Euclid III.35 (see Section 2.2. Book III) we would have $GV \cdot VP = QV^2$ (here we would need to extend $QV$ until it is a chord of the circle; see the inset in Figure 5.31).
Newton’s Lemma, continued 4

Proof (continued).

We now take the ellipse (along with the segments $GV$, $VP$, and $QV$) and stretch/shrink it in the direction of the blue “diameter” by a factor $1/c$, and stretch/contract it in the direction of the red “diameter” by a factor of $1/d$. This results in a circle of radius $1$. Since $GV$ and $VP$ lie along the blue diameter then these are transformed to the circle with lengths $GV/c$ and $VP/c$. Since $QV$ lies on a line parallel to the blue diameter then this is transformed to the circle with length $QV/d$. So from the above equation $GV \cdot VP = QV^2$ for the chords of a circle, we have

$$\frac{GV \cdot VP}{c^2} = \frac{QV^2}{d^2},$$

or:

$$(3) \quad VP = \frac{c^2}{GV} \cdot \frac{QV^2}{d^2}.$$

Newton’s Lemma, continued 5

Proof (continued).

Next, we express $VP$ in terms of $RQ$, and express $QV$ in terms of $QT$ (see the inset). Now triangle $XVP$ is similar to triangle $ECP$, because they share an angle at point $P$ and angles $PXV$ and $PEC$ are corresponding angles for parallel line segments $QV$ and $EF$ with transversal $CP$ (see Figure 1.7 center in Section 1.3. Properties of Angles); so the three angles of triangles $XVP$ and $ECP$ are the same. So $XP/VP = EP/CP$ or (since $EP = a$ by (5.55)) $XP/VP = a/c$, or:

$$(2) \quad XP = VP \cdot a/c.$$

Newton’s Lemma, continued 6

Proof (continued).

Triangle $QTX$ is similar to triangle $PFE$ since both of these are right triangles and angles $QXT$ and $PEF$ are alternate interior angles (or “parallel angles,” see Figure 1.7 let in Section 1.3. Properties of Angles) for parallel line segments $QX$ and $EF$ with transversal $EXE$; so the three angles of triangles $QTX$ and $PFE$ are the same. Also $QX/QT = PE/PE$ or (since $PF = h$) $QX/QT = a/h$, or:

$$(6) \quad QX = QT \cdot a/h.$$
Proof (continued). We now have:

\[ RQ \approx XP \text{ by (1)} \]
\[ = VP \cdot \frac{a}{c} \text{ by (2)} \]
\[ = \left( \frac{c^2}{GV} \cdot \frac{QV^2}{d^2} \right) \cdot \frac{a}{c} \text{ since by (3) } VP = \frac{c^2}{GV} \cdot \frac{QV^2}{d^2} \]
\[ \approx \frac{c^2}{2c} \cdot \frac{QV^2}{d^2} \cdot \frac{a}{c} \text{ since by (4) } GV \approx GP = 2c \]
\[ \approx \frac{c^2}{2c} \cdot \frac{QX^2}{d^2} \cdot \frac{a}{c} \text{ since by (5) } QV \approx QX \]
\[ = \frac{c^2}{2c} \cdot \left( \frac{QT \cdot a}{h} \right)^2 \cdot \frac{a}{c} \text{ since by (6) } QX = QT \cdot a/h \]
\[ = \frac{a^3}{2h^2d^2} \cdot QT^2. \]

Proof (continued). We now apply Apollonius VII.31, which is stated in Exercise 3.5.2 as: “All parallelograms circumscribed about any conjugate diameters of a given ellipse are equal.”

Notice that 1/4 of the areas of the inscribed rectangles above are \( hd \) (left) and \( ab \) (right). Since the areas of the parallelograms are the same by Apollonius VII.31, then \( hd = ab \).

Theorem 5.9. (Proposition 11 of Newton’s Principia Mathematica)

Theorem 5.9. (Proposition 11 of Newton’s Principia Mathematica.)
A body \( P \), orbiting according to Kepler 1 and 2 [i.e., Kepler’s 1st and 2nd Laws], moves under the effect of a centripetal force, directed to the centre \( S \), satisfying the law \( f = \frac{\text{Constant}}{r^2} \), where \( r \) is the distance \( SP \).

Proof. By equation (5.53) we have that the force \( f \) is proportional to \( RQ \). By Newton’s Lemma, \( RQ \) is approximately proportional to \( QT^2 \) (and so \( f \) is approximately proportional to \( QT^2 \)). The area of the triangle \( SPQ \) in Figure 5.30 right is \( SP \cdot QT/2 \) and this area is a constant for a fixed \( \Delta t \) by Kepler’s Second Law.
Theorem 5.9. (Proposition 11 of Newton's "Principia Mathematica," continued)

**Theorem 5.9.** (Proposition 11 of Newton’s *Principia Mathematica.*) A body $P$, orbiting according to Kepler 1 and 2 (i.e., Kepler’s 1st and 2nd Laws), moves under the effect of a centripetal force, directed to the centre $S$, satisfying the law $f = \frac{\text{Constant}}{r^2}$, where $r$ is the distance $SP$.

**Proof (continued).** Since $SP \cdot QT/2$ is constant, then $QT$ is inversely proportional to $SP$. Hence $f$ is approximately inversely proportional to $SP = r$. That is, $f \approx \frac{\text{Constant}}{r^2}$. By taking a limit as $\Delta t \to 0$, we have $Q \to P$ and the approximation becomes precise. □