History of Geometry

Chapter 5. Trigonometry

5.9. Trigonometric Formulation for Conics—Proofs of Theorems

Theorem 5.9.A

Theorem 5.9.A. With the parameters introduced above and in Figure 5.26 we have the relations

\[ r = \frac{p}{1 + e \cos \varphi} \quad \text{and} \quad r = a - ex = a - ae \cos u \]

where \( p \) is the vertical distance from a focus to the ellipse and \( x \) is the directed distance of \( P \) from the minor axis of the ellipse (when the ellipse has its major axis horizontal; see Figure 3.4 in Section 3.2. The Ellipse).

Proof. Recall that the sum of the distance of \( P \) to the two foci is twice the length of the major axis (this is equation (3.5) in Section 3.2. The Ellipse). So the distance \( BF \) is equal to \( a \). The lengths \( r/e \), \( p/e \), and \( a/e \) are given in Figure 3.4 (based on the definition of an ellipse in terms of a directrix and eccentricity), and also given in Figure 5.26.

Proof (continued).

From Figure 5.26 we see that distance \( p/e \) equals distance \( r/e \) plus the base of the right triangle with hypotenuse \( FP \). The base has length \( r \cos \varphi \) and so \( \frac{p}{e} = r \cos \varphi + \frac{r}{e} \). That is, \( p = er \cos \varphi + r = r(e \cos \varphi + 1) \) or \( r = \frac{p}{1 + e \cos \varphi} \), as claimed.

Also from Figure 5.26, distance \( a/e \) equals distance \( r/e \) plus the base of the triangle with hypotenuse \( OP' \). The base has length \( a \cos u \) and so \( \frac{a}{e} = \frac{r}{e} + a \cos u \). That is \( a = r + ea \cos u \), or \( r = a - ea \cos u \), as claimed. Also, \( x = a \cos u \) so we also have \( r = a - ex \), as claimed. \( \square \)
**Theorem 5.9.B**

**Theorem 5.9.B.** The area $\mathcal{A}$ swept out by the line joining the focus $F$ to a point $P$ on the ellipse over an angle $\varphi$ measured from the semimajor axis (see Figure 5.27, left) is

$$\mathcal{A} = \frac{ab}{2} (u - e \sin u).$$

**Proof.** We seek the shaded area $\mathcal{A}$ in Figure 5.27 (left).

![Figure 5.27](image)

We stretch the ellipse vertically by a factor of $a/b$ (that is, the $y$-value of each point is multiplied by $a/b$).

**Theorem 5.9.B (continued 2)**

**Theorem 5.9.B.** The area $\mathcal{A}$ swept out by the line joining the focus $F$ to a point $P$ on the ellipse over an angle $\varphi$ measured from the semimajor axis (see Figure 5.27, left) is

$$\mathcal{A} = \frac{ab}{2} (u - e \sin u).$$

**Proof (continued).** . . .

$$\mathcal{B} = \frac{a^2 u}{2} = \mathcal{T} = \frac{a^2 u}{2} - \frac{a^2 e \sin u}{2} = \frac{a^2}{2} (u - e \sin u),$$

and hence

$$\mathcal{A} = \left(\frac{b}{a}\right) \frac{a^2}{2} (u - e \sin u) = \frac{ab}{2} (u - e \sin u),$$

as claimed.

**Theorem 5.9.B (continued 1)**

**Proof (continued).**

With $\mathcal{B}$ as the shaded area in Figure 5.27 (right) we then have $\mathcal{B} = \frac{a^2}{b} \mathcal{A}$ (the idea is similar to that of Theorem 1.6, though that does not rigorously justify this claim). The area of the sector in the circle with central angle $u$ measured in radians is $a^2 u/2$. With $\mathcal{T}$ as the area of the triangle in Figure 5.27 (right), we have that $a^2 u/2$ is then $\mathcal{T} + \mathcal{B}$. Since $\mathcal{T} = \frac{1}{2}(ae)(a \sin u) = \frac{1}{2} a^2 e \sin u$. Therefore,

$$\mathcal{B} = \frac{a^2 u}{2} = \mathcal{T} = \frac{a^2 u}{2} - \frac{a^2 e \sin u}{2} = \frac{a^2}{2} (u - e \sin u), . . .$$