Section 1.7. The Pythagorean Theorem

Note. We saw in the previous section that the Pythagorean Theorem was known (at least in some cases) 3,600 to 4,000 years ago.

Note. Figure 1.15 gives a visual presentation of three proofs of the Pythagorean Theorem from three “civilizations,” Chinese, Indian, and Arabic. With $c$ as the length of the hypotenuse, we consider a square of area $c^2$ in Figure 1.15(a).

![Figure 1.15](image.png)

Note. In Figure 1.15(b), four right triangles with legs of lengths $a$ and $b$ and hypotenuse with length $c$ are introduced. Based on the fact that the angles of a triangle sum to $180^\circ$, we can show that Figure 1.15(b) actually is a square. Now we can use the fact that the area of a triangle is $1/2$(base)(altitude) (see Section 1.5. The Computation of Areas) to show that the area of the square of Figure 1.15(b) is $(a+b)^2 = 4 \times (\frac{1}{2}ab) + c^2$ or $a^2 + 2ab + b^2 = 2ab + c^2$ or $a^2 + b^2 = c^2$. The textbook credits this proof to Chou-pei Suan-ching of China in 250 BCE (the reference on
this is B.L. van der Waerden, *Geometry and Algebra in Ancient Civilization*, Berlin: Springer-Verlag (1983)). The triangles can also be sifted around to represent the same square of Figure 1.15(b) in terms of two square (of areas $a^2$ and $b^2$) and two $a$ by $b$ rectangles (see Figure 1.15, right).

**Note.** In Figure 1.15(c) four right triangles are *removed* from the square of area $c^2$. Again, since the sum of the angles of a triangle is $180^\circ$ then the triangles “fit” together to form the configuration of Figure 1.15(c). Computing areas gives $c^2 = 4 \times (\tfrac{1}{2}ab) + (a-b)^2$ or $c^2 = 2ab + (a^2 - 2ab+b^2)$ or $a^2 + b^2 = c^2$. The text book attributes this proof to the Indian Bhāskara II (1114–1185), but does not give a reference. A solid reference about this is Kim Plofker’s “Mathematics in India,” in Victor Katz (ed.), *The Mathematics of Egypt, Mesopotamia, China, India, and Islam: A Sourcebook*, Princeton University Press (2007) (see pages 476–477).

**Note.** Yet another, but similar, proof is illustrated in Figure 1.15(d). This is a combination of the two methods above, in that two right triangles with legs of lengths $a$ and $b$ are added to the outside of the square of area $c^2$ and two such right triangles are “subtracted” from inside the square. So the result (in white and light gray in Figure 1.15(d)) is the same as the area of the original square (namely, $c^2$) and equals $a^2 + b^2$ as can be seen in the figure (which is, again, justified by the fact that the angles of a triangle sum to $180^\circ$). This proof is attributed to Thābit ibn Qurra (828–901); see F. J. Swetz’s *From Five Fingers to Infinity*, Open Court (1996). A webpage with animations illustrating these three proofs, and many others, is available on the Many Proofs of Pythagorean Theorem webpage.
Note. Consider the figure below (the left part). We first give an informal argument for the Pythagorean Theorem. To each side of right triangle $\triangle AB\Gamma$ has been attached a square. We want to show that the area of square $\triangle B\Gamma E\Delta$ equals the sum of the areas of squares $\triangle ABZH$ and $\triangle A\Gamma K\Theta$. The areas of the two dark grey triangle $\triangle B\Delta A$ and $\triangle BZ\Gamma$ are the same, since one triangle can be obtained from the other by rotating through $90^\circ$ about point $B$. The triangle $\triangle BZ\Gamma$ has the same base and altitude as the square $\triangle BAHZ$ (the common base is the length of the segment $BZ$ and the common height is the length of segment $AB$). The triangle $\triangle B\Delta A$ has the same base and height as the rectangle $\triangle B\Delta \Lambda P$ (the common base is the length of line segment $B\Delta$ and the common height is the length of segment $\Delta \Gamma$). So half the area of square $\triangle ABZH$ equals half the area of rectangle $\triangle B\Delta \Lambda P$, and hence the area of $\triangle ABZH$ equals the area of rectangle $\triangle B\Delta \Lambda P$. Similarly, the area of square $\triangle A\Gamma K\Theta$ equals the area of rectangle $\triangle \Lambda E\Gamma P$. Since square $\triangle B\Gamma E\Delta$ is composed of the two rectangles $\triangle B\Delta \Lambda P$ and $\triangle \Lambda E\Gamma P$, then the sum of the areas of squares $\triangle ABZH$ and $\triangle A\Gamma K\Theta$ equals the area of square $\triangle B\Gamma E\Delta$, as needed.

Figure. Part of Figure 1.19 (left, with the label $P$ added) and a modification of it similar to the figure for the Pythagorean Theorem given in Euclid’s *Elements*. 
Note. The argument above is basically the argument made by Euclid in his *Elements*. This is such a historical result given in a historical reference, we now reproduce Euclid’s proof as stated in Thomas Heath’s *The Thirteen Books of Euclid’s Elements, translated from the text of Heiberg, with introduction and commentary*, Second Edition, Cambridge: Cambridge University Press (1926) (reprinted in 1956 by Dover Publications). A copy is online at bibotu.com (accessed 9/10/2021). The bold-faced items are references to other results in the *Elements*.

**Proposition I.47.** In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle.

**Proof.** Let $ABC$ be a right-angled triangle having the angle $BAC$ right;

I say that the square on $BC$ is equal to the squares on $BA$, $AC$.

For let there be described on $BC$ the square $BDEC$, and on $BA$, $AC$ the squares $GB$, $HC$; [I.46]

through $A$ let $AL$ be drawn parallel to either $BD$ or $CE$, and let $AD$, $FC$ be joined.

Then since each of the angles $BAC$, $BAG$ is right, it follows that with a straight line $BA$, and at the point $A$ on it, the two straight lines $AC$, $AG$ not lying on the same side make the adjacent angles equal to two right angles;

therefore $CA$ is in a straight line with $AG$. [I.14]

For the same reason $BA$ is also in a straight line with $AH$.

And, since the angel $DBC$ is equal to the angle $FBA$: for each is right: let the angle $ABC$ be added to each;

therefore the whole angle $DBA$ is equal to the whole angle $FBC$. [C.N.2]

And, since $DB$ is equal to $BC$, and $FB$ to $BA$, the two sides $AB$, $BD$ are equal
to the two sides $FB$, $BC$ respectively, and the angle $ABD$ is equal to the angle $FBC$

therefore the base $AD$ is equal to the base $FC$, and the triangle $ABD$ is equal to the triangle $FBC$. [I.4]

Now the parallelogram $BL$ is double of the triangle $ABD$, for they have the same base $BD$ and are in the same parallels $BD$, $AL$. [I.41]

And the square $GB$ is double of the triangle $FBC$, for they again have the same base $FB$ and are in the same parallels $FB$, $GC$. [I.41]

[But the doubles of equals are equal to one another.]

Therefore the parallelogram $BL$ is also equal to the square $GB$.

Similarly, if $AE$, $BK$ be joined, the parallelogram $CL$ can also be proved equal to the square $HC$;

therefore the whole square $BDEC$ is equal to the two squares $GB$, $HC$. [C.N.2]

And the square $BDEC$ is described on $BC$, and the squares $GB$, $HC$ on $BA$, $AC$.

Therefore the square on the side $BC$ is equal to the squares on the sides $BA$, $AC$.

Therefore, etc. Q. E. D.

\textbf{Note.} Vatican Manuscript Number 190 dates from the 10th century and contains the Books I to XII of \textit{Elements}, along with some other work. The text displays properties indicating that it is a more ancient version than others that survive. The image here (from the \textit{Greek Mathematics and its Modern Heirs} webpage) is from the page containing the proof of the Pythagorean Theorem.
**Note.** Consider the right triangle $ABC$ in Figure 1.20. Introducing the perpendicular to segment $AB$ which contains point $C$, we get that triangles $DBC$ and $CBA$ are similar and triangles $DAC$ and $CAB$ are similar. So corresponding sides have lengths that are in the same proportion (this is Thales’ Intercept Theorem, Theorem 1.1). Therefore

\[
\frac{a}{p} = \frac{c}{a} \quad \text{which implies } a^2 = pc, \quad \text{and}
\]

\[
\frac{b}{q} = \frac{c}{b} \quad \text{which implies } b^2 = qc.
\]

So $a^2 + b^2 = (pc) + (qc) = (p + q)c = c^2$ since $c = p + q$.

This proof is credited to Leonardo of Pisa (also known as Fibonacci, circa 1170–1250) and was given in his *Practica Geometria* in 1200.
Note. We now turn out attention to the proof given by Pythagoras himself. Quoting from Sir Thomas Heath’s *A History of Greek Mathematics*, Volume I, Clarendon Press (1921), which is still in print by Dover Publications and available for online reading from archive.org:

“The next question is, how was the theorem proved by Pythagoras or the Pythagoreans? Vitrivius says that Pythagoras first discovered the triangle (3, 4, 5), and doubtless the theorem was first suggested by the discovery that this triangle is right-angled; but this discovery probably came to Greece from Egypt. ... Two possible lines are suggested on which the general proof may have been developed. One is that of decomposing square and rectangular areas into squares, rectangles and triangles, and piecing them together again after the manner of Eucl., Book II; the isosceles right-angles triangle gives the most obvious case of this method. The other line is one depending upon proportions; and we have good reason for supposing that Pythagoras developed a theory of proportion. ... [Euclid] proved I.47 [the Pythagorean Theorem] by the methods of Book I instead of by proportions in order to get the proposition into Book I instead of Book VI [on proportions], to which it must have been relegated if the proof by proportions had been used. If, on the other hand, Pythagoras had proved it by means of the methods of Books I and II, it would hardly have been necessary for Euclid to devise a new proof of I.47. Hence it would appear most probably that Pythagoras would prove the proposition by means of his (imperfect) theory of proportions.”
Note. For Pythagoras’ proof by means of his “imperfect” theory of proportions, as Heath speculates, we consider the four shaded triangles in Figure 1.21. They have hypotheses of 1, \(a\), \(b\), and \(c\), and they are similar to each other by construction. If \(k\) denotes the area of the triangle with hypotenuse 1, then the other triangles have the areas as given in the figure, namely \(ka^2\), \(kb^2\), and \(kc^2\). This holds by Theorem 1.6 (which appears in the Elements as Euclid’s VI.19). Comparing the center and left triangles, we see that \(ka^2 + kb^2 = kc^2\), or \(a^2 + b^2 = c^2\) as needed.

A quick comment is in order to Heath’s use of the term “imperfect.” The Pythagorean theory of proportion only applies to commensurable magnitudes. That is, it is only valid for rational proportions. This is suggested in Figure 1.6 (of Section 1.2. Similar Figures) where the technique of constructing rational links is given.

Note. We now use the Pythagorean Theorem to find the radii of the incircle and the circumcircle (denoted \(\rho\) and \(R\), respectively) of a given regular \(n\)-gon with sides of length 1. In the case \(n = 3\), these two circles are given in Figure 1.22 (left).
In each case, we consider a line segment from the center of the circle to a vertex \( v \) of the \( n \)-gon; notice that such a line segment has length \( R \). Next, we introduce a line segment from the center of the circle to the midpoint of one of the edges of the \( n \)-gon that has vertex \( v \) as one of its endpoints; notice that such a line segment has length \( \rho \). The center of the circle, the vertex \( v \), and the midpoint of the line segment then form a right triangle with sides of lengths \( r \), \( \rho \), and \( 1/2 \) (the \( 1/2 \) resulting from bisecting and edge of the \( n \)-gon; see Figure 1.22 for the cases of \( n = 3 \) and \( n = 5 \)). So by The Pythagorean Theorem we have \( R^2 = \rho^2 + (1/2)^2 \), or \( \rho = \sqrt{R^2 - 1/4} \). In the case \( n = 3 \), we introduce the distance \( h \) given in Figure 1.22 (left). We know, also by the Pythagorean Theorem, that \( h = \sqrt{3}/2 \) (since we are dealing with a 30-60-90 triangle). Then

\[
\rho = h - R = \frac{\sqrt{3}}{2} - R \quad \text{and so} \quad \sqrt{R^2 - 1/4} = \frac{\sqrt{3}}{2} - R, \quad \text{or} \quad R^2 - 1/4 = 3/4 - \sqrt{3}R + R^2, \quad \text{or} \quad R = 1/\sqrt{3} = \sqrt{3}/3.
\]

Hence

\[
\rho = \sqrt{(\sqrt{3}/3)^2 - 1/4} = \sqrt{1/3 - 1/4} = \sqrt{1/12} = 1/(2\sqrt{3}) = \sqrt{3}/6.
\]

This is the first entry in Table 1.1 below. Similarly, in Figure 1.22 (right) we introduce the distance \( \ell \) and we see that the larger shaded triangle has sides of lengths, 1, \( \ell \), and \( \Phi/2 \) (see the “golden ratio” in Section 1.4. The Regular Pentagon; recall that \( \Phi^2 = \Phi + 1 \)). It is left as Exercise 1.7.A to show that \( R \) and \( \rho \) take on the values given in Table 1.1 in the case that \( n = 5 \).
Table 1.1. Radius of incircle ($\rho$) and radius of circumscribed circle ($R$) for regular polygons with side length 1

<table>
<thead>
<tr>
<th>$n$</th>
<th>$R$</th>
<th>$\rho$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>$R = \frac{\sqrt{3}}{3}$</td>
<td>$\rho = \frac{\sqrt{3}}{6}$</td>
</tr>
<tr>
<td>4</td>
<td>$R = \frac{\sqrt{2}}{2}$</td>
<td>$\rho = \frac{1}{2}$</td>
</tr>
<tr>
<td>5</td>
<td>$R = \frac{1}{\sqrt{3} - \Phi} = \frac{\sqrt{2} + \Phi}{\sqrt{5}}$</td>
<td>$\rho = \frac{\sqrt{3} + 4\Phi}{2\sqrt{5}}$</td>
</tr>
<tr>
<td>6</td>
<td>$R = 1$</td>
<td>$\rho = \frac{\sqrt{3}}{2}$</td>
</tr>
<tr>
<td>10</td>
<td>$R = \Phi$</td>
<td>$\rho = \frac{\sqrt{3} + 4\Phi}{2}$</td>
</tr>
</tbody>
</table>