Section 3.1. The Parabola.

Note. As described in the introduction to Chapter 3, Menaechmus introduced a parabola as an intersections of a plane with cone. We’ll see in Figure 3.2 (center) below that the plane must be parallel to some line on the cone which passes through the vertex of the cone (such a line is called a generator of the cone). The result we give (which is stated as a definition) is due to Pappus of Alexandria (circa 290–circa 350) and appeared in his Mathematical Collection, Book VII, Proposition 238. Though I cannot find an image of Pappus online (the images in these notes are all referenced, but their historical accuracy is questionable). However, some of his work is still in print, including Book VII.

Definition. Let \( d \) be a line, called the directrix, and \( F \) a point, called the focus, at distance \( p \) from the directrix. The locus of all points \( P \) that have the same given distance \( \ell \) from \( F \) as from \( f \) is called a parabola. See Figure 3.1 (left).
Note. The definition above is the same one used, for example, in Calculus 3 (MATH 2110). See my online notes for this class on Section 11.6. Conic Sections.

A parabola is symmetric with respect to a line perpendicular to the directrix which passes through the focus. This line is called the *axis* of the parabola and the point at which it intersects the parabola is the *vertex* of the parabola. We can derive an equation for a parabola using Figure 3.1 (left). Let point $P = (x, y)$ be on the parabola in the coordinate system with its origin at the vertex and with the $x$-axis horizontal. Notice for point $P$ in the figure, the distance $\ell$ from $P$ to $F$ is the same as the distance from $P$ to the directrix and this distance is $\ell = x + p/2$. Now in the right triangle with vertices $P$ and $F$ and hypotenuse given by the line segment $PF$, the Pythagorean Theorem gives that

$$\left(x - \frac{p}{2}\right)^2 + y^2 = \ell^2 = \left(x + \frac{p}{2}\right)^2 \quad \text{or} \quad x^2 - px + \frac{p^2}{4} + y^2 = x^2 + px + \frac{p^2}{4} \quad \text{or} \quad y^2 = px.$$

The value $2p$ is the length of a vertical line segment through the focus from one side of the parabola to the other (in Figure 3.1, left); this is called the *latus rectum* and value $p$ is the *semi latus rectum*. 

![Fig. 3.1. Definition and tangent of a parabola](image)
**Note.** In the next result, the intersection of a plane with a cone of Menaechmus is shown to give the same curve as the one given by Pappus’ definition. The proof we present is due to Germinal Dandelin (April 12, 1794–February 15, 1847) and presented in his “Memoir on some remarkable properties of the parabolic focale [i.e., oblique strophoid],” *Nouveaux mémoires de l’Académie royale des sciences et belles-lettres de Bruxelles* (in French), 2, 171-200 (1822).

**Theorem 3.1.** (Apollonius’ Proposition I.11 in *Treatise on Conic Sections*)
If a cone is cut by a plane that has the same slope as the generators of the cone, then the intersection is a parabola.

**Note.** We now consider lines tangent to a parabola. This is straightforward in calculus using derivatives. But here we give an argument based on Euclidean geometry and a construction of the tangent line that could be performed with a compass and straightedge.

![Fig. 3.1. Definition and tangent of a parabola](image)

Let $P$ be an arbitrary point on the parabola and let line $t$ be the bisector of the
angle $BPF$, as given in Figure 3.1 (center). For $Q$ another point on line $t$, we have the lengths of segments $BQ$ and $QF$ are the same since triangles $BOQ$ and $FPQ$ are congruent (by Side-Angle-Side, say). Now segment $QF$ is longer that the distance from $Q$ to the directrix line $d$, since $BQ$ is not orthogonal to $d$. So all points of line $t$ (other than point $P$) lie outside the parabola. Therefore, line $t$ is tangent to the parabola at point $P$.

**Note.** Euclid I.15 states: “If two straight lines cut one another, then they make the vertical angles equal to one another.” So if we extend the line segment $BP$ in Figure 3.1 (center), then we can use it to represent a ray of light approaching the parabola from the right. The Law of Reflection states that the angle of incidence equals the angle of reflection. So if the parabola is a reflective surface, the when a ray of light comes in from the right parallel to the axis then its angle of incidence will be $\alpha$ and so its angle of reflection will also be $\alpha$ (see Figure 3.1 (right). Then, as just argued, the ray of light will travel to the focus (following the path from point $P$ to point $F$). This can be shown using calculus as well (see *Thomas’ Calculus, Early Transcendentals*, 12th Edition, Exercise number 81 of Section 11.6. Conic Sections).