Leonard Euler

Happy 300th Birthday!

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ETSU Math Department Seminar

April 4, 2007

This is the transcript of the ETSU Math Department seminar given in commemoration of Léonard Euler's 300th birthday. The PowerPoint slides for the presentation are available as a webpage at `www.etsu.edu/math/gardner/euler/Euler-Seminar.html`. The bold faced markers indicate when each particular slide should be viewed in relation to the transcript.

[Slide 1]

INTRODUCTION

[**Slide 2**] Léonard Euler was the most prolific mathematician in history! [**Slide 3**] William Dunham, author of *Euler — The Master of Us All*, estimates that Euler wrote around 25,000 pages of math, physics, and other works [page 176]. An attempt has been underway since 1911 to publish his collected works under the title *Opera Omnia*. Dunham estimates that a printed copy of the final work will weigh about 300 pounds! [page 51 of *The Mathematical Universe*]

[**Slide 4**] Euler made advances in the areas of number theory, calculus, algebra, geometry, and complex numbers. In addition, he set the foundations for graph theory and the calculus of variations. [**Slide 5**] He gave us our modern definition of "function" and introduced notation which we use today without a second thought: $f(x)$ for a function, π for $3.14159\cdots$, e for the base of the natural logarithm, i for the square root of -1, and Σ for summation. He published some of the most influential textbooks of all times, and one popular level science book.

In this presentation, we give a brief biography of Léonard Euler and discuss three of his most famous results: (1) the summing of the p-series with $p = 2$, (2) the use of infinite products in analytic number theory, and (3) introduction of the "Euler line" to classical plane geometry.

[**Slide 6**]

BIOGRAPHY

[**Slide 7**] [This is based primarily on E.T. Bell's *Men of Mathematics* and W. Dunham's *Euler — The Master of Us All*]

[**Slide 8**] Léonard Euler was born 300 years ago on April 15, 1707 in Basel, Switzerland. [**Slide 9**] His father, Paul, had an interest in math and was a pupil of Jacob Bernoulli. [**Slide 10**] Paul Euler was a Calvinist pastor in the town of Riechen, Switzerland, and he intended for his son to follow the same career path. Léonard Euler entered the University of Basel in 1721 and studied theology and Hebrew. [**Slide 11**] His mathematical ability was quickly noticed by Johann Bernoulli [**Slide 12**] who tutored him. He also became friends with Daniel and Nicolaus Bernoulli, sons of Johann. [**Slide 13**] He received his master's degree in 1724. [**Slide 14**] In 1725, Daniel Bernoulli took a position in Russia's St. Petersburg Academy and in 1726 he invited Euler to join him. [**Slide 15**] Euler arrived in St. Petersburg in 1727. [**Slide 16**] He was officially invited as a member of the medical section. The day Euler arrived in Russia, Catherine I died. [**Slide 17**] In the resulting turmoil, Euler switched over to the physics section. For the next six years, Euler worked away. One reason for his immersion in research was his fear of the many spies in the oppressive Russian state.[**Slide 18**] In 1733, Daniel Bernoulli returned to Switzerland and Euler stepped in as the lead mathematician at the St. Petersburg Academy. [**Slide 19**]

At this time, Euler married Catharina Gsell, the daughter of a Swiss painter who was living in St. Petersburg. [**Slide 20**] Together, they had 13 children but only 5 survived to their 20s. [**Slide 21**] E.T. Bell in *Men of Mathematics* says that Euler "would often compose his memoirs with a baby in his lap while the older children played all about him. The ease with which he wrote the most difficult mathematics is incredible." [page 145]

[**Slide 22**] Euler's reputation was solidified in 1735 with his solution of the so called "Basel Problem." [**Slide 23**] This is the problem of solving the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$. It was originally posed in 1644 by Pietro Mengoli and brought to the attention of the mathematical community by Jacob Bernoulli in 1689. [**Slide 24**] Euler showed, to the surprise of many, that (quoting from Dunham's *Euler*, pages 45-46): [**Slide 25**] "Now, however, against all expectation I have found an elegant expression for the sum of the series $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} +$ etc., which depends on the quadrature of the circle... I have found that six times the sum of this series is equal to the square of the circumference of a circle whose diameter is 1." That is, Euler showed that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. Euler then started pouring out papers which were published in the journal of the St. Petersburg Academy.

In 1736 [**Slide 26**] he published his text *Mechanica*. Newton's *Principia Mathematica* laid the foundation of classical physics and celestial motion. However, Newton's arguments were based on geometric arguments (versus analytic ones). [**Slide 27**] E.T. Bell comments [page 147] that "Newton's *Principia* might have been written by Archimedes… [but in Euler's *Mechanica*] for the first time the full power of the calculus was directed against mechanics and the modern era in that basic science began."

[**Slide 28**] In 1738, things literally got darker for Euler. He started to loose sight in his right eye, probably the result of a severe infection he had at the time [Dunham, page *xxii*]. Nonetheless, Euler's productivity continued. [**Slide 29**] However, he was tiring of the oppressive circumstances in Russia and when, in 1741, Prussia's Frederick the Great [**Slide 30**] invited him to join the Berlin Academy, he moved his family there. They would remain there for 25 years.

[**Slide 31**] The Berlin years were as productive as the St. Petersburg years! In 1744 he set the stage for the area of math called the calculus of variations. He published his text *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes*. [**Slide 32**] In 1748 he published a two-volume work *Introductio in Analysis Infinitorium* (literally, 'Introduction to the Analysis of the Infinities"). The infinities the title refers to are: infinite series, infinite products, and continued fractions. The fourth definition in the book is that of *function* [**Slide 33**]: "A function of a variable quantity is an analytic expression composed in any way whatsoever of the variable quantity and numbers or constant quantities." [Blanton's translation, page 3] The translator refers to the material of this book as "precalculus" — not in the sense that *we* might think of precalculus, but in the sense that it does not deal with derivatives or integrals. It is also in this book that Euler states $e^{ix} = \cos x + i \sin x$. Math historian Carl Boyer said of this work [from Durham's

Mathematical Universe, page 52] [**Slide 34**]: "[It] is probably the most influential textbook of modern times. It is the work which made the function concept basic to mathematics. It popularized the definition of logarithms as exponents and the definitions of the trigonometric functions as ratios. It crystallized the distinction between algebraic and transcendental functions and between elementary and higher functions. It developed the use of polar coordinates and of the parametric representation of curves. Many of our commonplace notations are derived from it. In a word, the *Introductio* did for elementary analysis what the *Elements* of Euclid did for geometry."

In 1755 [**Slide 35**] he published a work entitled *Institutiones calculi differentialis* ("Foundations of Differential Calculus"). Some of the chapter titles include: [**Slide 36**]

- On the Infinite and the Infinitely Small
- On the Differentiation of Functions of Two or More Variables
- On Differential Equations

Now for a few comments about Euler and rigor. First, the analysis done by Euler was surprisingly...intuitive! His work lacks ϵ/δ (which were introduced by Cauchy). As we will see, some of his arguments are, by modern standards, mathematically "fishy'! [**Slide 37**] He throws around symbols like

- $\dfrac{1}{\infty} = 0$
- $1 + \dfrac{1}{3} + \dfrac{1}{5} + \dfrac{1}{7} + \cdots = 0.66215 + \dfrac{1}{2}\ln(\infty)$
- $\dfrac{1 - x^0}{0} = -\ln x.$

[**Slide 38**] William Dunham comments (pages 171–72): "[H]is mathematics did not always display the rigor and precision of today's, most particularly in his cavalier use of the infinite. These shortcomings have given ammunition to mathematicians who criticize his work as primitive, intuitive, and decidedly pre-modern. They have a point." Euler should be viewed as a ground breaking pioneer! It was left to others to take the rough trail to new territory which Euler cut and clean it up, pave it, and make it a smooth ride for future traveler's!

Euler completed one other noteworthy book while in Berlin. He corresponded with her in over 200 letters on physics, logic, and astronomy. [**Slide 39**] The letters were published as "Letters of Euler on Different Subjects in Natural Philosophy Addressed to a German Princess" This became a popular work and "remains to this day one of history's finest examples of popular science." [Dunham, page *xxv*]

[**Slide 40**] However, Frederick the Great for some reason became irate with Euler, even calling him "my cyclops" in reference to his total loss of sight in his right eye. [**Slide 41**] Catherine the Great came to power in Russia, and the political environment lightened. [**Slide 42**] In 1766, Euler returned to St. Petersburg, Russia where he remained for the rest of his life. [**Slide 43**] During 1768 to 1770 he published *Institutiones calculi integralic* in which he addressed integral calculus in 3 volumes. By 1771, his sight totally failed and he was blind. His wife died in 1773. [**Slide 44**] Despite these personal tragedies, Euler became even more productive! In 1775, he wrote an average of one math paper per week [Dunham page *xxvi*)! He married his first wife's half sister in 1776. [**Slide 45**] On September 18, 1783 Euler spent the morning with his grandchildren and then worked on some questions concerning the flight of balloons, inspired by the recent hot-air balloon flight in Paris of the Montgolfier brothers. In the late afternoon he suffered a massive hemorrhage and died. [Dunham page *xxviii*] [**Slide 46**] E.T. Bell romantically describes his death: A little later he asked that his grandson be brought in. While playing with the child and drinking tea he suffered a stroke. The pipe dropped from his hand, and with the words "I die," "Euler ceased to live and calculate."

[**Slide 47**]

SUMMING THE P-SERIES WITH $P = 2$

[**Slide 48**] We are all familiar with the MacLaurin series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^{2k-1}}{(2k-1)!}.$$

[**Slide 49**] We introduce a function $P(x) = \dfrac{\sin x}{x}$ for $x \neq 0$ (and $P(0) = 1$) where

$$\begin{aligned} P(x) = \frac{\sin x}{x} &= \frac{x - x^3/3! + x^5/5! - x^7/7! + \cdots}{x} \text{ if } x \neq 0 \\ &= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots . \end{aligned}$$

Now Euler wants to treat this series just like he would any polynomial, and factor it! This requires the use of an infinite product. Infinite products can be put on a rigorous foundation by considering (analogous to infinite series) "partial products" and limits. But this wasn't Euler's style! [**Slide 50**] Reasoning that $P(x) = 0$ whenever $\sin x = 0$ (except for $x = 0$), he factored $P(x)$ into

$$\begin{aligned} P(x) &= \left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right) \times \left(1 - \frac{x}{2\pi}\right)\left(1 - \frac{x}{2\pi}\right) \times \left(1 - \frac{x}{3\pi}\right)\left(1 - \frac{x}{3\pi}\right) \times \cdots \\ &= \left(1 - \frac{x^2}{\pi^2}\right) \times \left(1 - \frac{x^2}{4\pi^2}\right) \times \left(1 - \frac{x^2}{9\pi^2}\right) \times \cdots . \end{aligned}$$

Notice that *this* is the critical step! By using the series representation of $\sin x$, Euler has introduced both the squares of the natural numbers and π^2.

If we multiply out these factors and collect together powers of x, then we get

$$P(x) = 1 - \left(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \frac{1}{16\pi^2} + \cdots\right) x^2 + \cdots .$$

[**Slide 51**] Since we also have

$$P(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} - \cdots ,$$

then by equating the coefficients of x^2 we see that

$$\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \frac{1}{16\pi^2} - \cdots = \frac{1}{3!}$$

or

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}.$$

Even Euler himself seemed to be troubled by some of these manipulations, as evidenced by the fact that he gave other arguments for the equation. [Dunham page 48] Rigorous proofs exist today and use relationships between Bernoulli numbers and the Riemann zeta function. In fact, these rigorous derivations also use the representations of $\sin x$ as an infinite product, just as Euler did. (See, for example, *Special Functions for Engineers and Applied Mathematicians*, Larry Andrews, Macmillan Publishing, 1986, page 89.)

[**Slide 52**] By considering coefficients of higher powers of x, one can show that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} \text{ and } \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{645}.$$

In fact, Euler went as far as showing in 1744 [Dunham page 54]:

$$\sum_{n=1}^{\infty} \frac{1}{n^{26}} = \frac{2^{24}}{27!}(76977927\pi^{26}) = \frac{1315862}{11094481976030578125}\pi^{26}.$$

[**Slide 53**]

ANALYTIC NUMBER THEORY

A rather cryptic equation appears on one of the Euler tricentennial posters (as well as the cover of Dunham's book):

$$\sum_{k=1}^{\infty} \frac{1}{k} = \prod_{p} \frac{1}{1 - 1/p}.$$

Since the left hand side is the divergent harmonic series, what does this mean? Here is Euler's argument from *Opera Omnia*, Series 1, Volume 14 [Dunham pages 67-69].

[Slide 54] Let $x = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \cdots$. Then

$$\begin{aligned}\frac{1}{2}x = x - \frac{1}{2}x &= \left[1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \cdots\right] - \left[\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} \cdots\right] \\ &= 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} \cdots\end{aligned}$$

and

$$\frac{1}{3}\left[\frac{1}{2}x\right] = \frac{1}{3}\left[1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} \cdots\right] = \frac{1}{3} + \frac{1}{9} + \frac{1}{15} + \frac{1}{27} \cdots$$

or

$$\frac{1 \cdot 2}{2 \cdot 3}x = 1 + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \frac{1}{17} + \frac{1}{19} + \frac{1}{23} + \frac{1}{25} + \cdots$$

where the denominators are not divisible by 2 or by 3. Repeating for the next prime: **[Slide 55]**

$$\frac{1 \cdot 2}{2 \cdot 3}x - \frac{1}{5}\left[\frac{1 \cdot 2}{2 \cdot 3}x\right] = \left[1 + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} \cdots\right] - \left[\frac{1}{5} + \frac{1}{25} + \frac{1}{35} + \frac{1}{55} \cdots\right]$$

or

$$\frac{1 \cdot 2 \cdot 4}{2 \cdot 3 \cdot 5}x = 1 + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \frac{1}{17} + \frac{1}{23} + \frac{1}{29} + \cdots$$

where the denominators are not divisible by 2 or by 3 or by 5. So if the primes are listed as $p_1, p_2, p_3, p_4, \ldots$ then this process produces:

$$\frac{1 \cdot 2 \cdot 4 \cdot 6 \cdot 10 \cdot \cdots \cdot (p_n - 1)}{2 \cdot 3 \cdot 5 \cdot 11 \cdot \cdots \cdot p_n}x = 1 + \frac{1}{p_n} + (\text{sum of smaller terms})$$

where the "smaller terms are reciprocal of natural numbers which are not divisible by 2 or by 3 or by 5 of by 7... or by p_n. **[Slide 56]** Continuing the process (that is, taking a limit) Euler concludes

$$\frac{1 \cdot 2 \cdot 4 \cdot 6 \cdot 10 \cdot 12 \cdot 16 \cdot \cdots \cdot (p_n - 1) \cdot \cdots}{2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot \cdots \cdot p_n \cdot \cdots}x = 1$$

or

$$x = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \cdots = \frac{1 \cdot 2 \cdot 4 \cdot 6 \cdot 10 \cdot 12 \cdot 16 \cdot \cdots \cdot (p_n - 1) \cdot \cdots}{2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot \cdots \cdot p_n \cdot \cdots}.$$

The final relevant pattern is:

$$\begin{aligned}
\frac{1}{1-1/2} &= \frac{2}{1} \\
\left(\frac{1}{1-1/2}\right)\left(\frac{1}{1-1/3}\right) &= \frac{2\cdot 3}{1\cdot 2} \\
\left(\frac{1}{1-1/2}\right)\left(\frac{1}{1-1/3}\right)\left(\frac{1}{1-1/5}\right) &= \frac{2\cdot 3\cdot 5}{1\cdot 2\cdot 4} \\
&\vdots \\
\left(\frac{1}{1-1/2}\right)\left(\frac{1}{1-1/3}\right)\left(\frac{1}{1-1/5}\right)\cdots\left(\frac{1}{1-1/p_n}\right) &= \frac{2\cdot 3\cdot 5\cdot 7\cdot 11\cdot\cdots\cdot p_n}{1\cdot 2\cdot 4\cdot 6\cdot 10\cdot\cdots\cdot(p_n-1)} \\
&= = \prod_{p\in P_n}\frac{1}{1-1/p}
\end{aligned}$$

where $P_n = \{2, 3, 5, 7, 11, \ldots, , p_n\}$. [**Slide 57**] Therefore, Euler concludes

$$x = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots = \sum_{k=1}^{\infty}\frac{1}{k} = \frac{2\cdot 3\cdot 5\cdot 7\cdot\cdots}{1\cdot 2\cdot 4\cdot 6\cdot\cdots} = \prod_{p\in P}\frac{1}{1-1/p}$$

where P is the set of prime numbers.

[**Slide 58**] Dunham colorfully describes Euler's "proof" in this way: "with its repeated operations on divergent series, [it] is as porous as the Swiss cheese of Euler's homeland." [page 68]

[**Slide 59**] But what does it mean? Leopold Kronecker proved in 1876 that for $s > 1$,

$$\sum_{k=1}^{\infty}\frac{1}{k^s} = \prod_{p\in P}\frac{1}{1-1/p^s},$$

and so Euler's result can be interpreted as a limiting case of Kronecker's. [Dunham page 70]

The real significance of Euler's work is that he has linked together analytic ideas (series) with number theoretic ideas (prime numbers) and, in the process, set the stage for the development of analytic number theory.

[**Slide 60**]

GEOMETRY AND THE EULER LINE

Plane Euclidean geometry lay mostly dormant for about 2,000 years after Euclid. In 1767, Euler proved a new result concerning properties of triangles. In the 100 years that followed, geometry enjoyed a bit of a rebirth, of course with the introduction of projective and non-Euclidean geometry, but also with new results in classical Euclidean geometry, such as the introduction of the Poncelet/Brianchon circle.

[**Slide 61**] The *orthocenter* of a triangle is the intersection of the triangles three altitudes. The *centroid* is the intersection of the three lines which run from a vertex to the midpoint of the opposite side. The *circumcenter* is the center of the circle which passes through the three vertices of the circle.

[**Slide 62**] Euler proved that these three points lie on the same line, called the *Euler Line*.

[**Slide 63**]

CONCLUSION

In *Euler – The Master of Us All* W. Dunham comments (page *xvi*): "No student of literature would be satisfied with a mere synopsis of Hamlet. In like fashion, no mathematician should go through a career without meeting Euler face-to-face. To do otherwise suggests not only an indifference about the past but also, in some fundamental way, a genuine selfishness."

[**Slide 64**] Dartmouth College has created the *Euler Archive* which contains PDF files of the original versions of Euler's work: `http://www.math.dartmouth.edu/∼euler/`

[**Slide 65**] For those of us who only speak English, there are some translations of Euler's work, though not many. Included are *Introduction to Analysis of the Infinite*, *Foundations of Differential Calculus*, and *Elements of Algebra*.

[**Slide 66**]

References

1. L. Andrews, *Special Functions for Engineers and Applied Mathematicians*, MacMillan Pub., 1985.

2. E. T. Bell, *Men of Mathematics*, Simon and Schuster, 1937.

3. W. Dunham, *The Mathematical Universe*, Wiley, 1994.

4. W. Dunham, *Euler – The Master of Us All*, Math. Assoc. of Amer., 1999.

5. L. Euler, *Introduction to Analysis of the Infinite*, translated by J. Blanton, Springer-Verlag, 1988.

6. L. Euler, *Foundations of Differential Calculus*, translated by J. Blanton, Springer-Verlag, 2000.

[**Slide 67**] (This is the image that was on the ETSU Euler birthday cake.)