Decompositions of uniform complete directed multigraphs into each of the orientations of a 4-cycle. (English summary)


Let $D^\lambda_v$ denote the uniform complete directed multigraph on $v$ vertices with multiplicity $\lambda$. For $g$ a directed graph, a $g$-decomposition of $D^\lambda_v$ is a decomposition of the arc multiset of $D^\lambda_v$ into isomorphic copies of $g$. There are four orientations of a 4-cycle: the 4-circuit and the digraphs $X, Y$ and $Z$ with arc sets \{(a, b), (b, c), (c, d), (a, d)\}, \{(a, b), (b, c), (d, c), (a, d)\} and \{(a, b), (c, b), (c, d), (a, d)\}, respectively.


With an arbitrary $\lambda$, the authors establish necessary and sufficient conditions for the existence of the decomposition of $D^\lambda_v$ into each of the orientations of a 4-cycle: a 4-circuit decomposition of $D^\lambda_v$ exists if and only if $\lambda v(v - 1) \equiv 0 \pmod{4}$, except $v = 4$ and $\lambda$ odd; an $X$-decomposition of $D^\lambda_v$ exists if and only if $\lambda v(v - 1) \equiv 0 \pmod{4}$, except $v = 5$ and $\lambda = 1$; a $Y$-decomposition of $D^\lambda_v$ exists if and only if $\lambda v(v - 1) \equiv 0 \pmod{4}$, except $\lambda = 1$; and a $Z$-decomposition of $D^\lambda_v$ exists if and only if $\lambda v(v - 1) \equiv 0 \pmod{4}$, except $\lambda = 1$. The authors employ direct methods of construction, and their constructions lead to necessary and sufficient conditions for the existence of such decompositions which admit cyclic or rotational automorphisms.