Some Rotational Automorphisms Of Mendelsohn Triple And Quadruple Systems

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ABSTRACT. A Mendelsohn design of order $v$ with block size $n$ is said to be $k$-rotational if it admits an automorphism consisting of a fixed point and $k$ cycles each of length $(v - 1)/k$. It is said to be $k$-near-rotational if it admits an automorphism consisting of $w$ fixed points and $k$ cycles each of length $(v - w)/k$ where $w$ is the order of the smallest nontrivial Mendelsohn design with block size $n$. A Mendelsohn triple system is $k$-transrotational if it admits an automorphism consisting of a fixed point, a transposition and $k$ cycles each of length $(v - 3)/k$. The question of existence is addressed for $k$-transrotational and $k$-near-rotational Mendelsohn triple systems and for $k$-rotational and $k$-near-rotational Mendelsohn quadruple systems.

1 Introduction

A Mendelsohn design of order $v$ with block size $n$, denoted $MD(v, n)$, is an ordered pair $(V, B)$ where $V$ is a $v$-element set of points and $B$ is a collection of cyclically ordered $n$-tuples of distinct elements of $V$, called blocks, such that every ordered pair of distinct elements of $V$ occurs in exactly one block of $B$. A $MD(v, n)$ is equivalent to an arc-disjoint decomposition of the complete directed graph on $v$-vertices into $n$-circuits. It is, therefore, also equivalent to a balanced $n$-circuit design of order $v$ with $\lambda = 1$. For a survey of these relationships, see [3].

A $MD(v, 3)$ is also called a Mendelsohn triple system of order $v$, denoted $MTS(v)$, and a $MD(v, 4)$ is called a Mendelsohn quadruple system of order $v$, denoted $MQS(v)$. Nathan Mendelsohn introduced $MTS$ as a generalization of Steiner triple systems (briefly, STS) and proved that a $MTS(v)$ exists if and only if $v \equiv 0$ or $1 \pmod{3}$, $v \neq 6$ [12] (Mendelsohn called these structures "cyclic triple systems"; the term "Mendelsohn triple system" is due to Mathon and Rosa [11]). A $MQS(v)$ exists if and only if $v \equiv 0$ or $1 \pmod{4}$, $v \neq 4$ [1, 19]. The spectra of $MD(v, n)$s is now known for all $n$ such that $3 \leq n \leq 16$, $n \neq 15$ (see [1, 2, 4]).

An automorphism of a $MD(v, n)$ is a permutation of the point-set $V$ which fixes the set of blocks $B$. The orbit of a block under an automorphism $\pi$ is the collection of images of the block under the powers of $\pi$. A collection of blocks $\beta$ is a collection of base blocks for a $MD(v, n)$ under the automorphism $\pi$ if the orbits of the blocks of $\beta$ produce a set of blocks for a $MD(v, n)$ and exactly one block of $\beta$ occurs in each orbit. A permutation $\pi$ of a $v$-element set is of type $[\pi] = [\pi_1, \pi_2, \ldots, \pi_v]$ if the disjoint cyclic decomposition of $\pi$ consists of $\pi_i$ cycles of length $i$. It follows that $\sum i\pi_i = v$.

Several types of automorphisms have been studied for the question "For what values of $v$ does there exist a $STS(v)$ admitting an automorphism of the given type?" In particular, a $STS(v)$ admitting an automorphism of type $[0, 0, \ldots, 0, 1]$ is said to be cyclic and exists if and only if $v \equiv 1$ or $3 \pmod{6}$, $v \neq 9$ [15]. A $STS(v)$ which admits an automorphism of type $[1, 0, \ldots, 0, k, 0, \ldots, 0]$ is $k$-rotational. The spectra of $k$-rotational $STS$s are known for $k \in \{1, 2, 3, 4, 6\}$ [6,17]. This idea of rotational $STS$s has been extended somewhat. A $STS$ admitting an automorphism of type $[1, 1, 0, \ldots, 0, k; 0, \ldots, 0]$ is $k$-transrotational and the spectra of these are known for $k \in \{1, 2, 3\}$ [5, 10]. If a $STS$ admits an automorphism of type $[3, 0, \ldots, 0, k, 0, \ldots, 0]$, it is $k$-near-rotational and the spectra are known for $k \equiv 0, 2, 3, 4 \pmod{6}$ [9].

$MTS$s and $MQS$s have also been explored in connection with this type of question. A cyclic $MTS$ exists if and only if $v \equiv 1$ or $3 \pmod{6}$, $v \neq 9$ [8] and a cyclic $MQS(v)$ exists if and only if $v \equiv 1 \pmod{4}$ [14]. $k$-rotational $MTS$s are explored in [7] in which necessary and sufficient conditions are given for an infinite number of values of $k$, but not for all $k$. It is shown in [16] that a $1$-rotational $MQS(v)$ exists if and only if $v \equiv 1 \pmod{4}$. The purpose of this paper is to further explore the existence of $MTS$s and $MQS$s admitting various types of rotational automorphisms.

2 More Rotational Mendelsohn Quadruple Systems

Pennisi [16] has shown that a $1$-rotational $MQS(v)$ exists if and only if $v \equiv 1 \pmod{4}$. This result can be used to trivially construct a large class
of k-rotational MQSs.

Lemma 2.1. If \( v \equiv 1 \pmod{4} \) and \( v \equiv 1 \pmod{k} \), then there exists a k-rotational MQS(v)

Proof: If \( v \equiv 1 \pmod{4} \) then there is a 1-rotational MQS(v) admitting an automorphism \( \pi \) of type \([1, 0, \ldots, 0, 1, 0]\). If \( v \equiv 1 \pmod{k} \), then \( \pi^k \) is an automorphism of type \([1, 0, \ldots, 0, k, 0, \ldots, 0]\) and the MQS(v) is also k-rotational.

We now consider the case \( v \equiv 0 \pmod{4} \). We will let the point-set of a k-rotational MQS(v) be \( \{\infty\} \cup Z_N \times Z_k \) where \( N = \frac{v-1}{k} \). We will represent \((x, y) \in Z_N \times Z_k\) as \( x_y \) and let the relevant automorphism be \( \pi = (\infty)(0_0, 1_0, \ldots, (N-1)_0) \cdots (0_k, 1_k, \ldots, (N-1)_k) \). Here and throughout, we represent the ordered n-tuple containing the ordered pairs \((x_1, x_2), (x_2, x_3), \ldots, (x_{n-1}, x_n), (x_n, x_1)\) by any cyclic shift of \([x_1, x_2, \ldots, x_{n-1}, x_n]\)

Lemma 2.2. If \( v \equiv 4 \pmod{12} \) then there exists a 3-rotational MQS(v).

Proof: Let \( v = 12t + 4 \) and consider the collection of blocks:

\[
[\infty, 0_0, (3t)_1, (2t+1)_0, \infty, 0_1, (3t+1)_2, t_1], \infty, 0_2, (2t+1)_0, t_2],
\]

\[
[0_0, i_1, (1+2i)_2, (3+4i)_0] \text{ for } i = 0, 1, \ldots, 3t - 1, 3t + 1, \ldots, 4t \text{ and}
\]

\[
[0_1, (2+i)_0, (2+2i)_2, (3+4i)_1] \text{ for } i = 0, 1, \ldots, 3t - 1, 3t + 1, \ldots, 4t.
\]

These blocks, along with a collection of base blocks for a cyclic MQS(4t+1) on the point-set \( Z_N \times \{2\} \), where \( N = \frac{v-1}{3} \), under the automorphism \((0_0, 1_0, \ldots, (N-1)_2)\), form a collection of base blocks for a 3-rotational MQS(v).

As in Lemma 2.1, we can show that a k-rotational MQS(v) exists if \( v \equiv 0 \pmod{4} \), \( k \equiv 3 \pmod{6} \) and \( v \equiv 1 \pmod{k} \). Combining this fact with Lemma 2.1, we have:

Theorem 2.1. If \( k = 1 \) or \( k \) is even, then a k-rotational MQS(v) exists if and only if \( v \equiv 1 \pmod{4} \) and \( v \equiv 1 \pmod{k} \). If \( k \equiv 3 \pmod{6} \), then a k-rotational MQS(v) exists if and only if \( v \equiv 0 \) or \( 1 \pmod{4} \) and \( v \equiv 1 \pmod{k} \).

We leave the question of k-rotational MQS(v)s open for \( k \equiv 1 \) or \( 5 \pmod{6} \), \( k > 1 \) and \( v \equiv 0 \pmod{4} \).

3 Transrotational and Near-Rotational Mendelsohn Triple Systems

In this section we give necessary and sufficient conditions for the existence of a k-transrotational MTQ(v) for \( k \equiv 1, 2 \) or \( 3 \pmod{4} \) and for the existence
of a $k$-near-rotational $MTS(v)$ for all $k$. First we consider 1-transrotational $MTS(v)$ with the point-set $\{\infty, a, b\} \cup Z_N$ where the automorphism is $\pi = (\infty)(a, b)(0, 1, \ldots, (N - 1))$.

**Lemma 3.1.** If a $k$-transrotational $MTS(v)$ exists, then $v \equiv 3 \pmod{2k}$.

**Proof:** There must be some block of such a system of the form $[a, x, y]$ where $x, y \in Z_N \times Z_k$. Applying $\pi^N$ to this block, we see that $[\pi^N(a), x, y]$ is also a block of the system. So $\pi^N(a) = a$ and $N$ must be even and the result follows. \qed

**Lemma 3.2.** If $v \equiv 1$ or $3 \pmod{6}$ then there exists a 1-transrotational $MTS(v)$.

**Proof:** We consider three cases.

**case 1.** Suppose $v \equiv 1 \pmod{6}$, say $v = 6t + 1$. Consider the blocks:

\[
[\infty, 0, 3t - 1], [a, 0, 3t - 2], [b, 0, 3t], [\infty, a, b], \\
[0, (1 + i), (t + 1 + 2t)] \text{ for } i = 0, 1, \ldots, t - 2, \text{ and} \\
[0, (2t - 1 + i), (7t - 2 + 2i)] \text{ for } i = 0, 1, \ldots, t - 2.
\]

**case 2.** Suppose $v \equiv 3 \pmod{12}$, say $v = 12t + 3$. Consider the blocks:

\[
[\infty, 0, 2t], [a, 0, t], [b, 0, 3t], [\infty, a, b], [0, 4t, 8t], [0, 8t, 4t], \text{ and} \\
[0, (1 + i), (2t + 2 + 2i)] \text{ for } i = 0, 1, \ldots, t - 2, t, t + 1, \ldots, 2t - 2, \text{ and} \\
[0, (4t + 1 + i), (14t + 1 + 2i)] \text{ for } i = 0, 1, \ldots, 2t - 1.
\]

**case 3.** Suppose $v \equiv 9 \pmod{12}$, say $v = 12t + 9$. Consider the blocks:

\[
[\infty, 0, (2t + 1)], [a, 0, (5t + 3)], [b, 0, (11t + 5)], [\infty, a, b], \\
[0, (4t + 2), (8t + 4)], [0, (8t + 4), (4t + 2)], \\
[0, (1 + i), (2t + 3 + 2i)] \text{ for } i = 0, 1, \ldots, 2t - 1, \text{ and} \\
[0, (4t + 3 + i), (2t + 2 + 2i)] \text{ for } i = 0, 1, \ldots, t - 1, t + 1, t + 2, \ldots, 2t \\
\text{ (omitted if } t = 0 \text{ and let } i = 0, 2 \text{ if } t = 1).}
\]

In each case, the collection of blocks is a collection of base blocks for a 1-transrotational $MTS(v)$ under $\pi$. \qed

We now turn our attention to 2-transrotational $MTS(v)$ on the point-set $\{\infty, a, b\} \cup Z_k \times Z_2$ where $k = \frac{v - 3}{2}$ under the obvious automorphism. We will need the following structures. An $(A, n)$-system is a partitioning of the set $\{1, 2, \ldots, 2n\}$ into ordered pairs $(a_r, b_r)$ such that $b_r = a_r + r$ for $r = 1, 2, \ldots, n$. An $(A, n)$-system exists if and only if $n \equiv 0$ or $1 \pmod{4}$ [20]. A partitioning of the set $\{1, 2, \ldots, 2n - 1, 2n + 1\}$ into ordered pairs $(a_r, b_r)$ such that $b_r = a_r + r$ for $r = 1, 2, \ldots, n$ is called a $(B, n)$-system.
and exists if and only if \( n \equiv 2 \text{ or } 3 \pmod{4} \) [13]. A \((C, n)\)-system is a partitioning of the set \( \{1, 2, \ldots, n, n + 2, n + 3, \ldots, 2n + 1\} \) into distinct ordered pairs \((a_r, b_r)\) such that \( b_r = a_r + r \) for \( r = 1, 2, \ldots, n \). A \((C, n)\)-system exists if and only if \( n \equiv 0 \text{ or } 3 \pmod{4} \) [18]. A partitioning of the set \( \{1, 2, \ldots, n, n + 2, n + 3, \ldots, 2n, 2n + 2\} \) into ordered pairs \((a_r, b_r)\) such that \( b_r = a_r + r \) for \( r = 1, 2, \ldots, n \) is called a \((D, n)\)-system and exists if and only if \( n \equiv 1 \text{ or } 2 \pmod{4} \), \( n \neq 1 \) [18]. Notice that for each of these systems, the set \( \{r, a_r + r, b_r + r| r = 1, 2, \ldots, n\} \) includes all but two elements of the set \( \{1, 2, \ldots, 3n + 2\} \). It is this property of which we will take advantage.

**Lemma 3.3.** If \( v \equiv 7 \pmod{12} \) then there exists a 2-transrotational MTS\((v)\).

**Proof:** We consider four cases.

---

**case 1.** Suppose \( v \equiv 7 \text{ or } 31 \pmod{96} \), say \( v = 24t + 7 \) where \( t \equiv 0 \text{ or } 1 \pmod{4} \). Consider the blocks:

\[
\begin{align*}
&[\infty, a, b], [\infty, 0_0, 2_0], [\infty, 0_1, 2_1], \\
&[0_0, (i + 1)_{1}, (2i + 1)_{0}] \text{ for } i = 0, 1, \ldots, 6t, \\
&[0_1, (6t + 1 + i)_{0}, (1 + 2i)_{1}] \text{ for } i = 0, 1, \ldots, 6t, \\
&[0_0, (2r)_0, (2b_r + 2t)_0], [0_1, (2r)_1, (2b_r + 2t)_1], [(2b_r + 2t)_0, (2r)_0, 0_0], \\
&[(2b_r + 2t)_1, (2r)_1, 0_1] \text{ for } r = 2, 3, \ldots, t \text{ where the } b_r \text{ are from an } \\
&(A, t) - \text{system, and} \\
&a, 0_0, (2a_1 + 2t)_0], [a, 0_1, (2a_1 + 2t)_1], [b, (2b_1 + 2t)_0, 0_0], \\
&[b, (2b_1 + 2t)_1, 0_1], [(2b_1 + 2t)_0, 2_0, 0_0], [(2b_1 + 2t)_1, 2_1, 0_1] \text{ where } a_1 \\
&\text{and } b_1 \text{ are from the } (A, t) - \text{system used above.}
\]

---

**case 2.** Suppose \( v \equiv 55 \text{ or } 79 \pmod{96} \), say \( v = 24t + 7 \) where \( t \equiv 2 \text{ or } 3 \pmod{4} \). Consider the blocks:

\[
\begin{align*}
&[\infty, a, b], [\infty, 0_0, 2_0], [\infty, 0_1, 2_1], \\
&[0_0, (i + 1)_{1}, (2i + 1)_{0}] \text{ for } i = 0, 1, \ldots, 6t, \\
&[0_1, (6t + 1 + i)_{0}, (1 + 2i)_{1}] \text{ for } i = 0, 1, \ldots, 6t, \\
&[0_0, (2r)_0, (2b_r + 2t)_0], [0_1, (2r)_1, (2b_r + 2t)_1], [(2b_r + 2t)_0, (2r)_0, 0_0], \\
&[(2b_r + 2t)_1, (2r)_1, 0_1] \text{ for } r = 2, 3, \ldots, t \text{ where the } b_r \text{ are from a } \\
&(B, t) - \text{system, and} \\
&a, 0_0, (2a_1 + 2t)_0], [a, 0_1, (2a_1 + 2t)_1], [b, (2b_1 + 2t)_0, 0_0], \\
&[b, (2b_1 + 2t)_1, 0_1], [(2b_1 + 2t)_0, 2_0, 0_0], [(2b_1 + 2t)_1, 2_1, 0_1] \\
&\text{where } a_1 \text{ and } b_1 \text{ are from the } (B, t) - \text{system used above.}
\]

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case 3. Suppose \( v \equiv 19 \) or \( 43 \) (mod 96), say \( v = 24t + 19 \) where \( t \equiv 0 \) or 1 (mod 4). Consider the blocks:

\[
[\infty, a, b], [\infty, 0_0, (6t + 4)_0], [\infty, 0_1, (6t + 4)_1],
\]
\[
[0_0, (i + 1)_1, (2i + 1)_0] \text{ for } i = 0, 1, \ldots, 6t + 3,
\]
\[
[0_1, (6t + 4 + i)_0, (1 + 2i)_1] \text{ for } i = 0, 1, \ldots, 6t + 3,
\]
\[
[0_0, (2r)_0, (2b_r + 2t)_0], [0_1, (2r)_1, (2b_r + 2t)_1], [(2b_r + 2t)_0, (2r)_0, 0_0],
\]
\[
[(2b_r + 2t)_1, (2r)_1, 0_1] \text{ for } r = 1, 2, \ldots, t \text{ where the } b_r \text{ are from an } (A, t) - \text{system, and}
\]
\[
[a, 0_0, (6t + 2)_0], [a, 0_1, (6t + 2)_1], [b, 0_0, (6t + 6)_0], [b, 0_1, (6t + 6)_1].
\]

case 4. Suppose \( v \equiv 67 \) or \( 91 \) (mod 96), say \( v = 24t + 19 \) where \( t \equiv 2 \) or 3 (mod 4). Consider the blocks:

\[
[\infty, a, b], [\infty, 0_0, (6t + 4)_0], [\infty, 0_1, (6t + 4)_1],
\]
\[
[0_0, (i + 1)_1, (2i + 1)_0] \text{ for } i = 0, 1, \ldots, 6t + 3,
\]
\[
[0_1, (6t + 4 + i)_0, (1 + 2i)_1] \text{ for } i = 0, 1, \ldots, 6t + 3,
\]
\[
[0_0, (2r)_0, (2b_r + 2t)_0], [0_1, (2r)_1, (2b_r + 2t)_1], [(2b_r + 2t)_0, (2r)_0, 0_0],
\]
\[
[(2b_r + 2t)_1, (2r)_1, 0_1] \text{ for } r = 1, 2, \ldots, t \text{ where the } b_r \text{ are from a } (B, t) - \text{system, and}
\]
\[
[a, 0_0, (6t)_0], [a, 0_1, (6t)_1], [b, 0_0, (6t + 8)_0], [b, 0_1, (6t + 8)_1].
\]

In either case, the collection of blocks is a collection of base blocks for a 2-transrotational \( MTS(v) \) under \( \pi \).

\[\square\]

**Lemma 3.4.** If \( v \equiv 3 \) (mod 24) then there exists a 2-transrotational \( MTS(v) \).

**Proof:** Suppose \( v = 24t + 3 \). Consider the blocks:

\[
[\infty, a, b], [a, 0_0, (3t)_1], [a, 0_1, (3t)_0], [b, 0_0, (9t)_1], [b, 0_1, (9t)_0], [0_0, (8t)_0, (4t)_0],
\]
\[
[\infty, 0_1, ((t - 1)/2)_0 \text{ and } [\infty, 0_0, ((7t + 1)/2)_1] \text{ (omit these blocks if } t \text{ is even)}],
\]
\[
[\infty, 0_1, ((13t)/2)_0 \text{ and } [\infty, 0_0, ((19t)/2)_1] \text{ (omit these blocks if } t \text{ is odd)}],
\]
\[
[0_0, (1 + i)_0, (9t + 1 + 2i)_0] \text{ and } [0_1, (1 + i)_1, (9t + 1 + 2i)_1],
\]
\[
\text{for } i = 0, 1, \ldots, t - 1,
\]
\[
[0_0, (10t + i)_0, (9t + 2i)_0] \text{ and } [0_1, (10t + i)_1, (9t + 2i)_1] \text{ for } i = 0, 1, \ldots, t - 1,
\]
\[
[0_0, i_1, (9t - 1 - i)_1] \text{ for } i = 0, 1, \ldots, 3t - 1,
\]
\[
[0_0, (6t + 1 + i)_1, (3t - 1 - i)_1] \text{ for } i = 0, 1, \ldots, 3t - 2,
\]
\[
[0_1, i_0, (9t - 1 - i)_0] \text{ for } i = 0, 1, \ldots, 3t - 1 \text{ (omit } i = (t - 1)/2 \text{ if } t \text{ is odd), and}
\]
\[
[0_1, (6t + 1 + i)_0, (3t - 1 - i)_0] \text{ for } i = 0, 1, \ldots, 3t - 2 \text{(omit } i = (t - 2)/2 \text{ if } t \text{ is even)}.
\]

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This collection of blocks is a collection of base blocks for a 2-transrotational $MTS(v)$ under $\pi$. $\Box$

Lemma 3.5. If $v \equiv 15 \pmod{24}$ then there exists a 2-transrotational $MTS(v)$.

Proof: We consider four cases.

Case 1. Suppose $v = 15$. Consider the blocks:

$[\infty, a, b], [\infty, 0, 0, 1], [\infty, 0, 1, 40], [a, 0, 0, 1], [b, 0, 0, 5], [0, 0, 3, 1], [0, 1, 3, 1, 1], [0, 2, 0, 40], [0, 2, 1, 5], [0, 1, 2, 5],$ and $[0, 1, 0, 41].$

Case 2. Suppose $v = 39$. Consider the blocks:

$[\infty, a, b], [\infty, 0, 0, 1], [\infty, 0, 1, 10], [a, 0, 0, 140], [b, 0, 0, 160], [a, 0, 1, 41], [b, 0, 1, 161], [0, 0, 10, 70], [0, 0, 20, 90], [0, 0, 30, 80], [0, 0, 80, 30], [0, 0, 120, 60], [0, 0, 101, 10].$

$[0, 1, 161, 140],$

$[0, 1, 10, (2i + 1), 1]$ for $i = 0, 1, \ldots, 8$, and

$[0, 1, (11 + i), 0, (4 + 2i), 1]$ for $i = 0, 1, 2, 3, 4, 6.$

Case 3. Suppose $v \equiv 15$ or $87 \pmod{96}, v \geq 87$, say $v = 24t + 15$ where $t \equiv 0$ or $3 \pmod{4}$ and $t \geq 3$. Consider the blocks:

$[\infty, 0, (8t + 4), 0], [\infty, 0, 1, (8t + 4), 1],$

$[\infty, a, b], [a, 0, 0, 1], [a, 0, (2t + 2), 1], [b, 0, 1, (2t + 1), 0], [b, 0, 1, (12t + 5), 0],$

$[0, 1, 0, (4t + 3), 0], [0, 1, (4t + 2), 1, (8t + 4), 1], [0, 1, (6t + 2), 0, (12t + 5), 1],$

$[0, 0, (i + 1), 1, (2i + 1), 0]$ for $i = 0, 1, \ldots, 2t, 2t + 2, 2t + 3, \ldots, 6t + 1,$

$[0, 1, (6t + 3 + i), 0, (1 + 2i), 1]$ for $i = 0, 1, \ldots, 6t + 1$, and

$[0, 0, (2r), 0, (2b_r + 2t), 0], [0, 1, (2r), 1, (2b_r + 2t), 1], [(2b_r + 2t), 0, (2r), 0],$

$[(2b_r + 2t), 1, (2r), 1, 0]$ for $r = 1, 2, \ldots, t$ where the $b_r$ are from a $(C, t) -$ system.

Case 4. Suppose $v \equiv 39$ or $63 \pmod{96}, v \geq 63$, say $v = 24t + 15$ where $t \equiv 1$ or $2 \pmod{4}$ and $t \geq 2$. Consider the blocks:

$[\infty, 0, (8t + 4), 0], [\infty, 0, 1, (8t + 4), 1],$

$[\infty, a, b], [a, 0, 0, 1], [a, 0, (2t + 2), 1], [b, 0, 1, (2t + 1), 0], [b, 0, 1, (12t + 5), 0],$

$[0, 0, 1, (4t + 3), 0], [0, 1, (4t + 2), 1, (8t + 4), 1], [0, 1, (6t + 2), 0, (12t + 5), 1],$

$[0, 0, (i + 1), 1, (2i + 1), 0]$ for $i = 0, 1, \ldots, 2t, 2t + 2, 2t + 3, \ldots, 6t + 1,$

$[0, 1, (6t + 3 + i), 0, (1 + 2i), 1]$ for $i = 0, 1, \ldots, 6t + 1$, and

$[0, 0, (2r), 0, (2b_r + 2t), 0], [0, 1, (2r), 1, (2b_r + 2t), 1], [(2b_r + 2t), 0, (2r), 0],$

$[(2b_r + 2t), 1, (2r), 1, 0]$ for $r = 1, 2, \ldots, t$ where the $b_r$ are from a $(D, t) -$ system.
In each case, the collection of blocks is a collection of base blocks for a 2-transrotational \( MTS(v) \) under \( \pi \).

Notice that Lemmas 3.1 and 3.3-3.5 combine to tell us that a 2-transrotational \( MTS(v) \) exists if and only if \( v \equiv 3 \) or 7 (mod 12). By taking odd powers of automorphisms, the results of this section give:

**Theorem 3.1.** If \( k \equiv 1, 2, \text{ or } 3 \text{ (mod 4)} \) then a \( k \)-transrotational \( MTS(v) \) exists if and only if \( v \equiv 0 \) or 1 (mod 3) and \( v \equiv 3 \) (mod \( 2k \)).

We now consider near-rotational \( MTSs \). If \( \equiv 1 \) or 3 (mod 6) then a 1-near-rotational \( MTS(v) \) has the base block given in Lemma 3.2 under the automorphism \( (\infty)(a)(b)(0, 1, \ldots, v - 4) \). This fact along with the following lemma give the sufficient conditions for the existence of a 1-near-rotational \( MTS(v) \).

**Lemma 3.7.** If \( v \equiv 0 \) or 4 (mod 6), \( v \neq 12 \) then there exists a 1-near-rotational \( MTS(v) \).

**Proof:** If \( v \equiv 0 \) or 4 (mod 6), \( v \neq 12 \), then there exists a cyclic \( MTS(v-3) \). Let \( \beta \) be a set of base blocks for such a system on the point set \( Z_{v-3} \) under the automorphism \( (0, 1, \ldots, v - 4) \). With \( [x, y, z] \in \beta \), associate the differences \( \delta_1 = (y - x) \) (mod \( v - 3 \)), \( \delta_2 = (z - y) \) (mod \( v - 3 \)) and \( \delta_3 = (x - z) \) (mod \( v - 3 \)). Then it is necessary that \( \delta_1 + \delta_2 + \delta_3 \equiv 0 \) (mod \( v - 3 \)). If \( v \equiv 4 \) (mod 6), then \( \delta_1, \delta_2, \delta_3 \) are distinct. If \( v \equiv 0 \) (mod 6) then one block of \( \beta \) may have associated differences that satisfy \( \delta_1 = \delta_2 = \delta_3 = \frac{v-3}{3} \) and another block may have differences satisfying the condition \( \delta_1 = \delta_2 = \delta_3 = \frac{2(v-3)}{3} \). These two base blocks are said to be short orbit blocks since the lengths of their orbits are \( \frac{1}{3} \) the lengths of the orbits of any other base block of this system. To construct a 1-near-rotational \( MTS(v) \), consider the set \( \beta / \{b\} \) where \( b \) is any element of \( \beta \) other than a short orbit block. Let \( d_1, d_2, d_3 \) be the differences associated with \( b \). The set \( \beta \cup \{[\infty_1, \infty_2, \infty_3], [\infty_3, \infty_2, \infty_1], [\infty_1, 0, d_1], [\infty_2, 0, d_2], [\infty_3, 0, d_3]\} / \{b\} \) is a set of base blocks for a 1-near-rotational \( MTS(v) \) on \( \infty_1, \infty_2, \infty_3 \cup Z_{v-3} \) under the automorphism \( (\infty_1)(\infty_2)(\infty_3)(0, 1, \ldots, v - 4) \). 

A 1-near-rotational \( MTS(12) \) is equivalent to partitioning the set of differences \( \{1, 2, 4, 5, 7, 8\} \) (the differences 3 and 6 being associated with short orbit blocks) into two sets \( \{d_1, d_2, d_3\} \) and \( \{d_4, d_5, d_6\} \) such that \( d_1 + d_2 + d_3 \equiv d_4 + d_5 + d_6 \equiv 0 \) (mod 9). Clearly, this cannot be done and a 1-near-rotational \( MTS(12) \) does not exist.

By taking powers of the automorphism, the existence of 1-near-rotational \( MTSs \) gives us:

**Theorem 3.2.** A \( k \)-near-rotational \( MTS(v) \) exists if and only if \( v \equiv 0 \) or 1 (mod 3), \( v \neq 6 \) and \( v \equiv 3 \) (mod \( k \)) and if \( k = 1 \) then \( v \neq 12 \).
Proof: We need only to present a 3-near-rotational $MTS(12)$. Consider the blocks:

$[\infty_1, \infty_2, \infty_3], [\infty_3, \infty_2, \infty_1], [\infty_1, 0_0, 2_0], [\infty_1, 0_1, 2_1], [\infty_1, 0_2, 2_2], [\infty_2, 0_0, 1_0], [\infty_2, 0_1, 2_0], [\infty_2, 0_2, 0_0], [\infty_3, 0_1, 0_0], [\infty_3, 0_2, 0_1], [\infty_3, 0_0, 0_2], [0_0, 1_0, 2_0], [0_1, 1_1, 2_1], [0_2, 1_2, 2_2], [0_0, 1_1, 2_2], [2_2, 1_1, 0_0], [0_0, 2_1, 1_2], and [1_2, 2_1, 0_0].$

This is a collection of base blocks for a 3-near-rotational $MTS(12)$ on the point-set $\{\infty_1, \infty_2, \infty_3\} \cup \{0, 1, 2\} \times \{0, 1, 2\}$ under the obvious automorphism.

4 Near-Rotational Mendelsohn Quadruple Systems

In general, we say that a $MD(v, n)$ is $k$-near-rotational if it admits an automorphism consisting of $w$ fixed points and $k$ cycles of length $\frac{v-w}{k}$ where $w$ is the order of the smallest nontrivial $MD(v, n)$. It is fairly easy to see that the fixed points of an automorphism of a $MD(v, n)$ form a subsystem and so by having an automorphism with $w$ fixed points and $k$ cycles of the same length, we are as "near" as possible to having a $k$-rotational $MD(v, n)$. Therefore, with $n = 4$ we say that a $MQS(v)$ is $k$-near-rotational if it admits an automorphism consisting of 5 fixed points and $k$ cycles of length $\frac{v-5}{k}$. In this section we give necessary and sufficient conditions for the existence of $k$-near-rotational $MQS$s for all $k$.

We consider 1-near-rotational $MQS(v)$ on the point-set $\{\infty_1, \infty_2, \ldots, \infty_5\} \cup \mathbb{Z}_{v-5}$ under the obvious automorphism.

Lemma 4.1. If $v \equiv 0 \pmod{4}$, $v \geq 16$ then there exists a 1-near-rotational $MQS(v)$.

Proof: Suppose $v = 4t$. Consider the blocks:

$[\infty_1, 0, (2t - 7), (4t - 13)], [\infty_2, 0, (2t - 5), (4t - 9)], [\infty_3, 0, (2t - 3), (4t - 3)], [\infty_4, 0, (2t - 1), (4t - 3)], [\infty_5, 0, (2t + 1), (4t + 3)],$ and $[0, (2i + 1), (4i + 3), (4t - 3 + 2i)]$ for $i = 0, 1, \ldots, t - 5$

(omit these blocks if $t = 4$).

These blocks along with the blocks for a $MQS(5)$ on the points $\{\infty_1, \infty_2, \ldots, \infty_5\}$ form a collection of base blocks for a 1-near-rotational $MQS(v)$ under the given automorphism.

Lemma 4.2. If $v \equiv 1 \pmod{4}$, $v \geq 17$ then there exists a 1-near-rotational $MQS(v)$.
Proof: We consider three cases.

Case 1. Suppose \( v = 17 \). Consider the blocks:

\[
[\infty_1, 0, 1, 3], [\infty_2, 0, 4, 9], [\infty_3, 0, 6, 1], [\infty_4, 0, 8, 5], [\infty_5, 0, 10, 9],
\]

and \([0, 3, 6, 9]\).

Case 2. Suppose \( v \equiv 1 \pmod{8} \), say \( v = 8t + 1 \) where \( t \geq 3 \). Consider the blocks:

\[
[\infty_1, 0, (4t - 5), (8t - 9)], [\infty_2, 0, (4t - 3), (8t - 5)], [\infty_3, 0, (4t - 1), (8t - 1)],
\]

\[
[\infty_4, 0, (4t + 1), (6t + 1)], [\infty_5, 0, (6t - 3), (4t - 3)], [0, (2t - 1), (4t - 2),
\]

\( (6t - 3) \) and

\[0, (2i + 1), (4i + 3), (8t - 2 + 2i) \] for \( i = 0, 1, \ldots, t - 2, t, t + 1, \ldots, 2t - 4 \).

Case 3. Suppose \( v \equiv 5 \pmod{8} \), say \( v = 8t + 5 \) where \( t \geq 2 \). Consider the blocks:

\[
[\infty_1, 0, (4t - 3), (8t - 5)], [\infty_2, 0, (4t - 1), (8t - 1)], [\infty_3, 0, (4t + 1), (8t + 3)],
\]

\[
[\infty_4, 0, (4t + 3), (6t + 2)], [\infty_5, 0, (6t + 1), (4t + 1)], [0, 2t, 4t, 6t] \text{ and}
\]

\[0, (2i + 1), (4i + 3), (8t + 2 + 2i) \] for \( i = 0, 1, \ldots, t - 2, t, t + 1, \ldots, 2t - 3 \).

In both cases, these blocks along with the blocks for a \( MQS(5) \) on the point-set \( \{\infty_1, \ldots, \infty_5\} \) form a collection of base blocks for a 1-near-rotational \( MQS(v) \) under the given automorphism.

Clearly, a \( k \)-near-rotational \( MQS(v) \) does not exist for \( v < 16 \). Therefore, as in the previous sections, Lemmas 4.1 and 4.2 give us:

**Theorem 4.1.** A \( k \)-near-rotational \( MQS(v) \) exists if and only if \( v \equiv 0 \) or \( 1 \pmod{4} \), \( v \geq 16 \) and \( v \equiv 5 \pmod{k} \).

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**References**


