Some Rotational Automorphisms Of Mendelsohn Triple And Quadruple Systems

Gary D. Coker
Department of Mathematics
Williamsburg Technical College
Kingstree, South Carolina 29556-4197
and
Robert B. Gardner
Department of Mathematics
East Tennesse State University
Johnson City, Tennessee 37614-0663

ABSTRACT. A Mendelsohn design of order v with block size n is said to be k-rotational if it admits an automorphism consisting of a fixed point and k cycles each of length $(v-1)/k$. It is said to be k-near-rotational if it admits an automorphism consisting of w fixed points and k cycles each of length $(v-w)/k$ where w is the order of the smallest nontrivial Mendelsohn design with block size n. A Mendelsohn triple system is k-transrotational if it admits an automorphism consisting of a fixed point, a transposition and k cycles each of length $(v-3)/k$. The question of existence is addressed for k-transrotational and k-near-rotational Mendelsohn triple systems and for k-rotational and k-near-rotational Mendelsohn quadruple systems.

1 Introduction

A *Mendelsohn design* of order v with block size n, denoted $MD(v,n)$, is an ordered pair (V, B) where V is a v-element set of *points* and B is a collection of cyclically ordered n-tuples of distinct elements of V, called *blocks*, such that every ordered pair of distinct elements of V occurs in exactly one block of B. A $MD(v,n)$ is equivalent to an arc-disjoint decomposition of the complete directed graph on v-vertices into n-circuits. It is, therefore, also equivalent to a balanced n-circuit design of order v with $\lambda=1$. For a survey of these relationships, see [3].

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A $MD(v,3)$ is also called a *Mendelsohn triple system* of order v, denoted $MTS(v)$, and a $MD(v,4)$ is called a *Mendelsohn quadruple system* of order v, denoted $MQS(v)$. Nathan Mendelsohn introduced MTSs as a generalization of Steiner triple systems (briefly, STS) and proved that a $MTS(v)$ exists if and only if $v \equiv 0$ or $1 \pmod 3$, $v \neq 6$ [12] (Mendelsohn called these structures "cyclic triple systems"; the term "Mendelsohn triple system" is due to Mathon and Rosa [11]). A $MQS(v)$ exists if and only if $v \equiv 0$ or $1 \pmod 4$, $v \neq 4$ [1, 19]. The spectra of $MD(v,n)$s is now known for all n such that $3 \leq n \leq 16$, $n \neq 15$ (see [1, 2, 4]).

An *automorphism* of a $MD(v,n)$ is a permutation of the point-set V which fixes the set of blocks B. The *orbit* of a block under an automorphism π is the collection of images of the block under the powers of π. A collection of blocks β is a *collection of base blocks* for a $MD(v,n)$ under the automorphism π if the orbits of the blocks of β produce a set of blocks for a $MD(v,n)$ and exactly one block of β occurs in each orbit. A permutation π of a v-element set is of *type* $[\pi] = [\pi_1, \pi_2, \ldots, \pi_v]$ if the disjoint cyclic decomposition of π consists of π_i cycles of length i. It follows that $\sum i\pi_i = v$.

Several types of automorphisms have been studied for the question "For what values of v does there exist a $STS(v)$ admitting an automorphism of the given type?" In particular, a $STS(v)$ admitting an automorphism of type $[0, 0, \ldots, 0, 1]$ is said to be *cyclic* and exists if and only if $v \equiv 1$ or $3 \pmod 6$, $v \neq 9$ [15]. A $STS(v)$ which admits an automorphism of type $[1, 0, \ldots, 0, k, 0, \ldots, 0]$ is *k-rotational.* The spectra of k-rotational STSs are known for $k \in \{1, 2, 3, 4, 6\}$ [6,17]. This idea of rotational STSs has been extended somewhat. A STS admitting an automorphism of type $[1, 1, 0, \ldots, 0, k, 0, \ldots, 0]$ is *k-transrotational* and the spectra of these are known for $k \in \{1, 2, 3\}$ [5, 10]. If a STS admits an automorphism of type $[3, 0, \ldots, 0, k, 0, \ldots, 0]$, it is *k-near-rotational* and the spectra are known for $k \equiv 0, 2, 3, 4 \pmod 6$ [9].

MTSs and MQSs have also been explored in connection with this type of question. A cyclic MTS exists if and only if $v \equiv 1$ or $3 \pmod 6$, $v \neq 9$ [8] and a cyclic $MQS(v)$ exists if and only if $v \equiv 1 \pmod 4$ [14]. k-rotational MTSs are explored in [7] in which necessary and sufficient conditions are given for an infinite number of values of k, but not for all k. It is shown in [16] that a 1-rotational $MQS(v)$ exists if and only if $v \equiv 1 \pmod 4$. The purpose of this paper is to further explore the existence of MTSs and MQSs admitting various types of rotational automorphisms.

2 More Rotational Mendelsohn Quadruple Systems

Pennisi [16] has shown that a 1-rotational $MQS(v)$ exists if and only if $v \equiv 1 \pmod 4$. This result can be used to trivially construct a large class of k-rotational MQSs.

Lemma 2.1. *If $v \equiv 1 \pmod 4$ and $v \equiv 1 \pmod k$, then there exists a k-rotational $MQS(v)$*

Proof: If $v \equiv 1 \pmod 4$ then there is a 1-rotational $MQS(v)$ admitting an automorphism π of type $[1, 0, \ldots, 0, 1, 0]$. If $v \equiv 1 \pmod k$, then π^k is an automorphism of type $[1, 0, \ldots, 0, k, 0, \ldots, 0]$ and the $MQS(v)$ is also k-rotational. □

We now consider the case $v \equiv 0 \pmod 4$. We will let the point-set of a k-rotational $MQS(v)$ be $\{\infty\} \cup Z_N \times Z_k$ where $N = \frac{v-1}{k}$. We will represent $(x, y) \in Z_N \times Z_k$ as x_y and let the relevant automorphism be $\pi = (\infty)(0_0, 1_0, \ldots, (N-1)_0) \ldots (0_{k-1}, 1_{k-1}, \ldots, (N-1)_{k-1})$. Here and throughout, we represent the ordered n-tuple containing the ordered pairs $(x_1, x_2), (x_2, x_3), \ldots, (x_{n-1}, x_n), (x_n, x_1)$ by any cyclic shift of $[x_1, x_2, \ldots, x_{n-1}, x_n]$

Lemma 2.2. *If $v \equiv 4 \pmod{12}$ then there exists a 3-rotational $MQS(v)$.*

Proof: Let $v = 12t + 4$ and consider the collection of blocks:

$$[\infty, 0_0, (3t)_1, (2t+1)_0], [\infty, 0_1, (3t+1)_2, t_1], [\infty, 0_2, (2t+1)_0, t_2],$$
$$[0_0, i_1, (1+2i)_2, (3+4i)_0] \text{ for } i = 0, 1, \ldots, 3t-1, 3t+1, \ldots, 4t \text{ and}$$
$$[0_1, (2+i)_0, (2+2i)_2, (3+4i)_1] \text{ for } i = 0, 1, \ldots, 3t-1, 3t+1, \ldots, 4t.$$

These blocks, along with a collection of base blocks fc- a cyclic $MQS(4t+1)$ on the point-set $Z_N \times \{2\}$, where $N = \frac{v-1}{3}$, under the automorphism $(0_2, 1_2, \ldots, (N-1)_2)$, form a collection of base blocks for a 3-rotational $MQS(v)$. □

As in Lemma 2.1, we can show that a k-rotational $MQS(v)$ exists if $v \equiv 0 \pmod 4$, $k \equiv 3 \pmod 6$ and $v \equiv 1 \pmod k$. Combining this fact with Lemma 2.1, we have:

Theorem 2.1. *If $k = 1$ or k is even, then a k-rotational $MQS(v)$ exists if and only if $v \equiv 1 \pmod 4$ and $v \equiv 1 \pmod k$. If $k \equiv 3 \pmod 6$, then a k-rotational $MQS(v)$ exists if and only if $v \equiv 0$ or $1 \pmod 4$ and $v \equiv 1 \pmod k$.*

We leave the question of k-rotational $MQS(v)$s open for $k \equiv 1$ or $5 \pmod 6$, $k > 1$ and $v \equiv 0 \pmod 4$.

3 Transrotational and Near-Rotational Mendelsohn Triple Systems

In this section we give necessary and sufficient conditions for the existence of a k-transrotational $MTS(v)$ for $k \equiv 1, 2$ or $3 \pmod 4$ and for the existence of a k-near- rotational $MTS(v)$ for all k. First we consider 1-transrotational $MTS(v)$ with the point-set $\{\infty, a, b\} \cup Z_N$ where the automorphism is $\pi = (\infty)(a, b)(0, 1, \ldots, (N-1))$.

Lemma 3.1. *If a k-transrotational $MTS(v)$ exists, then $v \equiv 3 \pmod{2k}$.*

Proof: There must be some block of such a system of the form $[a, x, y]$ where $x, y \in Z_N \times Z_k$. Applying π^N to this block, we see that $[\pi^N(a), x, y]$ is also a block of the system. So $\pi^N(a) = a$ and N must be even and the result follows. □

Lemma 3.2. *If $v \equiv 1$ or $3 \pmod 6$ then there exists a 1-transrotational $MTS(v)$.*

Proof: We consider three cases.

case 1. Suppose $v \equiv 1 \pmod 6$, say $v = 6t+1$. Consider the blocks:

$$[\infty, 0, 3t-1], [a, 0, 3t-2], [b, 0, 3t], [\infty, a, b],$$
$$[0, (1+i), (t+1+2i)] \text{ for } i = 0, 1, \ldots, t-2, \text{ and}$$
$$[0, (2t-1+i), (7t-2+2i)] \text{ for } i = 0, 1, \ldots, t-2.$$

case 2. Suppose $v \equiv 3 \pmod{12}$, say $v = 12t+3$. Consider the blocks:

$$[\infty, 0, 2t], [a, 0, t], [b, 0, 3t], [\infty, a, b], [0, 4t, 8t], [0, 8t, 4t], \text{ and}$$
$$[0, (1+i), (2t+2+2i)] \text{ for } i = 0, 1, \ldots, t-2, t, t+1, \ldots, 2t-2, \text{ and}$$
$$[0, (4t+1+i), (14t+1+2i)] \text{ for } i = 0, 1, \ldots, 2t-1.$$

case 3. Suppose $v \equiv 9 \pmod{12}$, say $v = 12t+9$. Consider the blocks:

$$[\infty, 0, (2t+1)], [a, 0, (5t+3)], [b, 0, (11t+5)], [\infty, a, b],$$
$$[0, (4t+2), (8t+4)], [0, (8t+4), (4t+2)],$$
$$[0, (1+i), (2t+3+2i)] \text{ for } i = 0, 1, \ldots, 2t-1, \text{ and}$$
$$[0, (4t+3+i), (2t+2+2i)] \text{ for } i = 0, 1, \ldots, t-1, t+1, t+2, \ldots, 2t$$
$$\text{(omit if } t = 0 \text{ and let } i = 0, 2 \text{ if } t = 1).$$

In each case, the collection of blocks is a collection of base blocks for a 1-transrotational $MTS(v)$ under π. □

We now turn our attention to 2-transrotational $MTS(v)$ on the point-set $\{\infty, a, b\} \cup Z_k \times Z_2$ where $k = \frac{v-3}{2}$ under the obvious automorphism. We will need the following structures. An (A, n)-*system* is a partitioning of the set $\{1, 2, \ldots, 2n\}$ into ordered pairs (a_r, b_r) such that $b_r = a_r + r$ for $r = 1, 2, \ldots, n$. An (A, n)-system exists if and only if $n \equiv 0$ or $1 \pmod 4$ [20]. A partitioning of the set $\{1, 2, \ldots, 2n-1, 2n+1\}$ into ordered pairs (a_r, b_r) such that $b_r = a_r + r$ for $r = 1, 2, \ldots, n$ is called a (B, n)-*system* and exists if and only if $n \equiv 2$ or 3 (mod 4) [13]. A (C,n)-*system* is a partitioning of the set $\{1,2,\ldots,n,n+2,n+3,\ldots,2n+1\}$ into distinct ordered pairs (a_r,b_r) such that $b_r = a_r + r$ for $r = 1,2,\ldots,n$. A (C,n)-system exists if and only if $n \equiv 0$ or 3 (mod 4) [18]. A partitioning of the set $\{1,2,\ldots,n,n+2,n+3,\ldots,2n,2n+2\}$ into ordered pairs (a_r,b_r) such that $b_r = a_r + r$ for $r = 1,2,\ldots,n$ is called a (D,n)-system and exists if and only if $n \equiv 1$ or 2 (mod 4), $n \neq 1$ [18]. Notice that for each of these systems, the set $\{r, a_r + r, b_r + r | r = 1,2,\ldots,n\}$ includes all but two elements of the set $\{1,2,\ldots,3n+2\}$. It is this property of which we will take advantage.

Lemma 3.3. *If* $v \equiv 7$ (mod 12) *then there exists a 2-transrotational* $MTS(v)$.

Proof: We consider four cases.
case 1. Suppose $v \equiv 7$ or 31 (mod 96), say $v = 24t + 7$ where $t \equiv 0$ or 1 (mod 4). Consider the blocks:

$[\infty, a, b], [\infty, 0_0, 2_0], [\infty, 0_1, 2_1]$,
$[0_0, (i+1)_1, (2i+1)_0]$ for $i = 0,1,\ldots,6t$,
$[0_1, (6t+1+i)_0, (1+2i)_1]$ for $i = 0,1,\ldots,6t$,
$[0_0, (2r)_0, (2b_r+2t)_0], [0_1, (2r)_1, (2b_r+2t)_1], [(2b_r+2t)_0, (2r)_0, 0_0]$, $[(2b_r+2t)_1, (2r)_1, 0_1]$ for $r = 2,3,\ldots,t$ where the b_r are from an (A,t) – system, and
$[a, 0_0, (2a_1+2t)_0], [a, 0_1, (2a_1+2t)_1], [b, (2b_1+2t)_0, 0_0]$, $[b, (2b_1+2t)_1, 0_1], [(2b_1+2t)_0, 2_0, 0_0], [(2b_1+2t)_1, 2_1, 0_1]$ where a_1 and b_1 are from the (A,t) – system used above.

case 2. Suppose $v \equiv 55$ or 79 (mod 96), say $v = 24t + 7$ where $t \equiv 2$ or 3 (mod 4). Consider the blocks:

$[\infty, a, b], [\infty, 0_0, 2_0], [\infty, 0_1, 2_1]$,
$[0_0, (i+1)_1, (2i+1)_0]$ for $i = 0,1,\ldots,6t$,
$[0_1, (6t+1+i)_0, (1+2i)_1]$ for $i = 0,1,\ldots,6t$,
$[0_0, (2r)_0, (2b_r+2t)_0], [0_1, (2r)_1, (2b_r+2t)_1], [(2b_r+2t)_0, (2r)_0, 0_0]$, $[(2b_r+2t)_1, (2r)_1, 0_1]$ for $r = 2,3,\ldots,t$ where the b_r are from a (B,t) – system, and
$[a, 0_0, (2a_1+2t)_0], [a, 0_1, (2a_1+2t)_1], [b, (2b_1+2t)_0, 0_0]$, $[b, (2b_1+2t)_1, 0_1], [(2b_1+2t)_0, 2_0, 0_0], [(2b_1+2t)_1, 2_1, 0_1]$ where a_1 and b_1 are from the (B,t) – system used above.

case 3. Suppose $v \equiv 19$ or $43 \pmod{96}$, say $v = 24t + 19$ where $t \equiv 0$ or $1 \pmod 4$. Consider the blocks:

$[\infty, a, b], [\infty, 0_0, (6t+4)_0], [\infty, 0_1, (6t+4)_1]$,
$[0_0, (i+1)_1, (2i+1)_0]$ for $i = 0, 1, \ldots, 6t+3$,
$[0_1, (6t+4+i)_0, (1+2i)_1]$ for $i = 0, 1, \ldots, 6t+3$,
$[0_0, (2r)_0, (2b_r+2t)_0], [0_1, (2r)_1, (2b_r+2t)_1], [(2b_r+2t)_0, (2r)_0, 0_0]$,
$[(2b_r+2t)_1, (2r)_1, 0_1]$ for $r = 1, 2, \ldots, t$ where the b_r are from an (A, t) – system, and
$[a, 0_0, (6t+2)_0], [a, 0_1, (6t+2)_1], [b, 0_0, (6t+6)_0], [b, 0_1, (6t+6)_1]$.

case 4. Suppose $v \equiv 67$ or $91 \pmod{96}$, say $v = 24t + 19$ where $t \equiv 2$ or $3 \pmod 4$. Consider the blocks:

$[\infty, a, b], [\infty, 0_0, (6t+4)_0], [\infty, 0_1, (6t+4)_1]$,
$[0_0, (i+1)_1, (2i+1)_0]$ for $i = 0, 1, \ldots, 6t+3$,
$[0_1, (6t+4+i)_0, (1+2i)_1]$ for $i = 0, 1, \ldots, 6t+3$,
$[0_0, (2r)_0, (2b_r+2t)_0], [0_1, (2r)_1, (2b_r+2t)_1], [(2b_r+2t)_0, (2r)_0, 0_0]$,
$[(2b_r+2t)_1, (2r)_1, 0_1]$ for $r = 1, 2, \ldots, t$ where the b_r are from a (B, t) – system, and
$[a, 0_0, (6t)_0], [a, 0_1, (6t)_1], [b, 0_0, (6t+8)_0], [b, 0_1, (6t+8)_1]$.

In either case, the collection of blocks is a collection of base blocks for a 2-transrotational $MTS(v)$ under π. □

Lemma 3.4. *If $v \equiv 3 \pmod{24}$ then there exists a 2-transrotational $MTS(v)$.*

Proof: Suppose $v = 24t + 3$. Consider the blocks:

$[\infty, a, b], [a, 0_0, (3t)_1], [a, 0_1, (3t)_0], [b, 0_0, (9t)_1], [b, 0_1, (9t)_0], [0_0, (8t)_0, (4t)_0]$,
$[\infty, 0_1, ((t-1)/2)_0]$ and $[\infty, 0_0, ((7t+1)/2)_1]$ (omit these blocks if t is even),
$[\infty, 0_1, ((13t)/2)_0]$ and $[\infty, 0_0, ((19t)/2)_1]$ (omit these blocks if t is odd),
$[0_0, (1+i)_0, (9t+1+2i)_0]$ and $[0_1, (1+i)_1, (9t+1+2i)_1]$
for $i = 0, 1, \ldots, t-1$,
$[0_0, (10t+i)_0, (9t+2i)_0]$ and $[0_1, (10t+i)_1, (9t+2i)_1]$ for $i = 0, 1, \ldots, t-1$,
$[0_0, i_1, (9t-1-i)_1]$ for $i = 0, 1, \ldots, 3t-1$,
$[0_0, (6t+1+i)_1, (3t-1-i)_1]$ for $i = 0, 1, \ldots, 3t-2$,
$[0_1, i_0, (9t-1-i)_0]$ for $i = 0, 1, \ldots, 3t-1$ (omit $i = (t-1)/2$ if t is odd), and
$[0_1, (6t+1+i)_0, (3t-1-i)_0]$ for $i = 0, 1, \ldots, 3t-2$ (omit $i = (t-2)/2$ if t is even).

This collection of blocks is a collection of base blocks for a 2-transrotational $MTS(v)$ under π. □

Lemma 3.5. *If* $v \equiv 15 \pmod{24}$ *then there exists a 2-transrotational* $MTS(v)$.

Proof: We consider four cases.

case 1. Suppose $v = 15$. Consider the blocks:

$[\infty, a, b], [\infty, 0_0, 0_1], [\infty, 0_1, 4_0], [a, 0_0, 1_1], [b, 0_0, 5_1], [a, 0_1, 1_0], [b, 0_1, 5_0],$
$[0_0, 3_0, 1_0], [0_1, 3_1, 1_1], [0_0, 2_0, 4_0], [0_0, 2_1, 5_0], [0_1, 2_0, 5_1]$, and $[0_1, 0_0, 4_1]$.

case 2. Suppose $v = 39$. Consider the blocks:

$[\infty, a, b], [\infty, 0_0, 0_1], [\infty, 0_1, 10_0], [a, 0_0, 14_0], [b, 0_0, 16_0], [a, 0_1, 4_1], [b, 0_1, 16_1],$
$[0_0, 1_0, 7_0], [0_0, 2_0, 9_0], [0_0, 3_0, 8_0], [0_0, 8_0, 3_0], [0_0, 12_0, 6_0], [0_0, 10_1, 1_0],$
$[0_0, 16_1, 14_0],$
$[0_1, i_0, (2i+1)_1]$ for $i = 0, 1, \ldots, 8$, and
$[0_1, (11+i)_0, (4+2i)_1]$ for $i = 0, 1, 2, 3, 4, 6$.

case 3. Suppose $v \equiv 15$ or $87 \pmod{96}$, $v \geq 87$, say $v = 24t + 15$ where $t \equiv 0$ or $3 \pmod 4$ and $t \geq 3$. Consider the blocks:

$[\infty, 0_0, (8t+4)_0], [\infty, 0_1, (8t+4)_1],$
$[\infty, a, b], [a, 0_0, 0_1], [a, 0_0, (2t+2)_1], [b, 0_1, (2t+1)_0], [b, 0_1, (12t+5)_0],$
$[0_0, 1_0, (4t+3)_0], [0_1, (4t+2)_1, (8t+4)_1], [0_1, (6t+2)_0, (12t+5)_1],$
$[0_0, (i+1)_1, (2i+1)_0]$ for $i = 0, 1, \ldots, 2t, 2t+2, 2t+3, \ldots, 6t+1,$
$[0_1, (6t+3+i)_0, (1+2i)_1]$ for $i = 0, 1, \ldots, 6t+1$, and
$[0_0, (2r)_0, (2b_r+2t)_0], [0_1, (2r)_1, (2b_r+2t)_1], [(2b_r+2t)_0, (2r)_0, 0_0],$
$[(2b_r+2t)_1, (2r)_1, 0_1]$ for $r = 1, 2, \ldots, t$ where the b_r are from a (C, t) − system.

case 4. Suppose $v \equiv 39$ or $63 \pmod{96}$, $v \geq 63$, say $v = 24t + 15$ where $t \equiv 1$ or $2 \pmod 4$ and $t \geq 2$. Consider the blocks:

$[\infty, 0_0, (8t+4)_0], [\infty, 0_1, (8t+4)_1],$
$[\infty, a, b], [a, 0_0, 0_1], [a, 0_0, (2t+2)_1], [b, 0_1, (2t+1)_0], [b, 0_1, (12t+5)_0],$
$[0_0, 1_0, (4t+3)_0], [0_1, (4t+2)_1, (8t+4)_1], [0_1, (6t+2)_0, (12t+5)_1],$
$[0_0, (i+1)_1, (2i+1)_0]$ for $i = 0, 1, \ldots, 2t, 2t+2, 2t+3, \ldots, 6t+1,$
$[0_1, (6t+3+i)_0, (1+2i)_1]$ for $i = 0, 1, \ldots, 6t+1$, and
$[0_0, (2r)_0, (2b_r+2t)_0], [0_1, (2r)_1, (2b_r+2t)_1], [(2b_r+2t)_0, (2r)_0, 0_0],$
$[(2b_r+2t)_1, (2r)_1, 0_1]$ for $r = 1, 2, \ldots, t$ where the b_r are from a (D, t) − system.

In each case, the collection of blocks is a collection of base blocks for a 2-transrotational $MTS(v)$ under π. □

Notice that Lemmas 3.1 and 3.3-3.5 combine to tell us that a 2-transrotational $MTS(v)$ exists if and only if $v \equiv 3$ or $7 \pmod{12}$. By taking odd powers of automorphisms, the results of this section give:

Theorem 3.1. *If $k \equiv 1, 2,$ or $3 \pmod 4$ then a k-transrotational $MTS(v)$ exists if and only if $v \equiv 0$ or $1 \pmod 3$ and $v \equiv 3 \pmod{2k}$.*

We now consider near-rotational MTSs. If $v \equiv 1$ or $3 \pmod 6$ then a 1-near-rotational $MTS(v)$ has the base block given in Lemma 3.2 under the automorphism $(\infty)(a)(b)(0, 1, \ldots, v-4)$. This fact along with the following lemma give the sufficient conditions for the existence of a 1-near-rotational $MTS(v)$.

Lemma 3.7. *If $v \equiv 0$ or $4 \pmod 6$, $v \neq 12$ then there exists a 1-near-rotational $MTS(v)$.*

Proof: If $v \equiv 0$ or $4 \pmod 6$, $v \neq 12$, then there exists a cyclic $MTS(v-3)$. Let β be a set of base blocks for such a system on the point set Z_{v-3} under the automorphism $(0, 1, \ldots, v-4)$. With $[x, y, z] \in \beta$, associate the *differences* $\delta_1 = (y-x) \pmod{v-3}$, $\delta_2 = (z-y) \pmod{v-3}$ and $\delta_3 = (x-z) \pmod{v-3}$. Then it is necessary that $\delta_1 + \delta_2 + \delta_3 \equiv 0 \pmod{v-3}$. If $v \equiv 4 \pmod 6$, then δ_1, δ_2 and δ_3 are distinct. If $v \equiv 0 \pmod 6$ then one block of β may have associated differences that satisfy $\delta_1 = \delta_2 = \delta_3 = \frac{v-3}{3}$ and another block may have differences satisfying the condition $\delta_1 = \delta_2 = \delta_3 = \frac{2(v-3)}{3}$. These two base blocks are said to be *short orbit* blocks since the lengths of their orbits are $\frac{1}{3}$ the lengths of the orbits of any other base block of this system. To construct a 1-near-rotational $MTS(v)$, consider the set $\beta/\{b\}$ where b is any element of β other than a short orbit block. Let d_1, d_2, d_3 be the differences associated with b. The set $\beta \cup \{[\infty_1, \infty_2, \infty_3], [\infty_3, \infty_2, \infty_1], [\infty_1, 0, d_1], [\infty_2, 0, d_2], [\infty_3, 0, d_3]\}/\{b\}$ is a set of base blocks for a 1-near-rotational $MTS(v)$ on $\{\infty_1, \infty_2, \infty_3\} \cup Z_{v-3}$ under the automorphism $(\infty_1)(\infty_2)(\infty_3)(0, 1, \ldots, v-4)$. □

A 1-near-rotational $MTS(12)$ is equivalent to partitioning the set of differences $\{1, 2, 4, 5, 7, 8\}$ (the differences 3 and 6 being associated with short orbit blocks) into two sets $\{d_1, d_2, d_3\}$ and $\{d_4, d_5, d_6\}$ such that $d_1 + d_2 + d_3 \equiv d_4 + d_5 + d_6 \equiv 0 \pmod 9$. Clearly, this cannot be done and a 1-near-rotational $MTS(12)$ does not exist.

By taking powers of the automorphism, the existence of 1-near-rotational MTSs gives us:

Theorem 3.2. *A k-near-rotational $MTS(v)$ exists if and only if $v \equiv 0$ or $1 \pmod 3$, $v \neq 6$ and $v \equiv 3 \pmod k$ and if $k = 1$ then $v \neq 12$.*

Proof: We need only to present a 3-near-rotational $MTS(12)$. Consider the blocks:

$$[\infty_1, \infty_2, \infty_3], [\infty_3, \infty_2, \infty_1], [\infty_1, 0_0, 2_0], [\infty_1, 0_1, 2_1], [\infty_1, 0_2, 2_2]$$
$$[\infty_2, 0_0, 0_1], [\infty_2, 0_1, 0_2], [\infty_2, 0_2, 0_0], [\infty_3, 0_1, 0_0], [\infty_3, 0_2, 0_1], [\infty_3, 0_0, 0_2],$$
$$[0_0, 1_0, 2_0], [0_1, 1_1, 2_1], [0_2, 1_2, 2_2], [0_0, 1_1, 2_2], [2_2, 1_1, 0_0], [0_0, 2_1, 1_2],$$
and $[1_2, 2_1, 0_0]$.

This is a collection of base blocks for a 3-near-rotational $MTS(12)$ on the point-set $\{\infty_1, \infty_2, \infty_3\} \cup \{0, 1, 2\} \times \{0, 1, 2\}$ under the obvious automorphism. □

4 Near-Rotational Mendelsohn Quadruple Systems

In general, we say that a $MD(v, n)$ is k-near-rotational if it admits an automorphism consisting of w fixed points and k cycles of length $\frac{v-w}{k}$ where w is the order of the smallest nontrivial $MD(v, n)$. It is fairly easy to see that the fixed points of an automorphism of a $MD(v, n)$ form a subsystem and so by having an automorphism with w fixed points and k cycles of the same length, we are as "near" as possible to having a k-rotational $MD(v, n)$. Therefore, with $n = 4$ we say that a $MQS(v)$ is k-near-rotational if it admits an automorphism consisting of 5 fixed points and k cycles of length $\frac{v-5}{k}$. In this section we give necessary and sufficient conditions for the existence of k-near-rotational MQSs for all k.

We consider 1-near-rotational $MQS(v)$ on the point-set $\{\infty_1, \infty_2, \ldots, \infty_5\} \cup Z_{v-5}$ under the obvious automorphism.

Lemma 4.1. *If $v \equiv 0 \pmod 4$, $v \geq 16$ then there exists a 1-near-rotational $MQS(v)$.*

Proof: Suppose $v = 4t$. Consider the blocks:

$$[\infty_1, 0, (2t-7), (4t-13)], [\infty_2, 0, (2t-5), (4t-9)],$$
$$[\infty_3, 0, (2t-3), (4t-3)], [\infty_4, 0, (2t-1), (4t-3)],$$
$$[\infty_5, 0, (2t+1), (4t+3)], \text{ and}$$
$$[0, (2i+1), (4i+3), (4t-3+2i)] \text{ for } i = 0, 1, \ldots, t-5$$
(omit these blocks if $t = 4$).

These blocks along with the blocks for a $MQS(5)$ on the points $\{\infty_1, \infty_2, \ldots, \infty_5\}$ form a collection of base blocks for a 1-near-rotational $MQS(v)$ under the given automorphism. □

Lemma 4.2. *If $v \equiv 1 \pmod 4$, $v \geq 17$ then there exists a 1-near-rotational $MQS(v)$.*

Proof: We consider three cases.
case 1. Suppose $v = 17$. Consider the blocks:

$$[\infty_1, 0, 1, 3], [\infty_2, 0, 4, 9], [\infty_3, 0, 6, 1], [\infty_4, 0, 8, 5], [\infty_5, 0, 10, 9], \text{ and } [0, 3, 6, 9].$$

case 2. Suppose $v \equiv 1 \pmod 8$, say $v = 8t + 1$ where $t \geq 3$. Consider the blocks:

$[\infty_1, 0, (4t-5), (8t-9)], [\infty_2, 0, (4t-3), (8t-5)], [\infty_3, 0, (4t-1), (8t-1)], [\infty_4, 0, (4t+1), (6t+1)], [\infty_5, 0, (6t-3), (4t-3)], [0, (2t-1), (4t-2), (6t-3)]$ and
$[0, (2i+1), (4i+3), (8t-2+2i)]$ for $i = 0, 1, \ldots, t-2, t, t+1, \ldots, 2t-4$.

case 3. Suppose $v \equiv 5 \pmod 8$, say $v = 8t + 5$ where $t \geq 2$. Consider the blocks:

$[\infty_1, 0, (4t-3), (8t-5)], [\infty_2, 0, (4t-1), (8t-1)], [\infty_3, 0, (4t+1), (8t+3)], [\infty_4, 0, (4t+3), (6t+2)], [\infty_5, 0, (6t+1), (4t+1)], [0, 2t, 4t, 6t]$ and
$[0, (2i+1), (4i+3), (8t+2+2i)]$ for $i = 0, 1, \ldots, t-2, t, t+1, \ldots, 2t-3$.

In both cases, these blocks along with the blocks for a $MQS(5)$ on the point-set $\{\infty_1, \ldots, \infty_5\}$ form a collection of base blocks for a 1-near-rotational $MQS(v)$ under the given automorphism. □

Clearly, a k-near-rotational $MQS(v)$ does not exist for $v < 16$. Therefore, as in the previous sections, Lernmas 4.1 and 4.2 give us:

Theorem 4.1. *A k-near-rotational $MQS(v)$ exists if and only if $v \equiv 0$ or $1 \pmod 4$, $v \geq 16$ and $v \equiv 5 \pmod k$.*

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