Decompositions of the Complete Digraph and the Complete Graph which admit Certain Automorphisms

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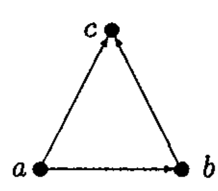
Abstract. Decompositions of the complete digraph into isomorphic copies of orientations of 4-cycles and 5-cycles are considered. Necessary and sufficient conditions for such decompositions admitting cyclic or rotational automorphisms are given. Some new decompositions of the complete digraph into certain (non-self-converse) orientations of the 6-cycle are also given. The spectrum of 4-, 6-, and 8-cycle systems admitting either a reverse automorphism or a k–rotational automorphism for any k are determined.

1 Introduction

Let D_v denote the complete digraph on v vertices. If g is a digraph, then a g*–decomposition* of D_v is a set $\gamma = \{g_1, g_2, \ldots, g_n\}$ of arc-disjoint subgraphs of D_v each of which is isomorphic to g and such that $\bigcup_{i=1}^{n} A(g_i) = A(D_v)$, where $A(G)$ is the arc set of digraph G. Several of these decompositions are equivalent to block designs. For example, a D_3–decomposition of D_v is equivalent to a Steiner triple system of order v. A k–circuit decomposition of D_v is equivalent to a k–Mendelsohn design of order v, denoted $M(k, v)$.

There are two orientations of the 3-cycle: the 3-circuit and the digraph (called a "transitive triple"):



A decomposition of D_v into 3-circuits is equivalent to a Mendelsohn triple system of order v (or a $M(3, v)$) which exists if and only if $v \equiv 0$ or 1 (mod 3), $v \neq 6$ [14]. A

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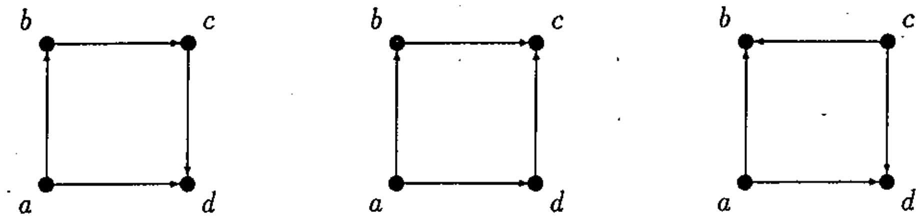
decomposition of D_v into transitive triples is equivalent to a directed triple system of order v, which exists if and only if $v \equiv 0$ or 1 (mod 3) [11].

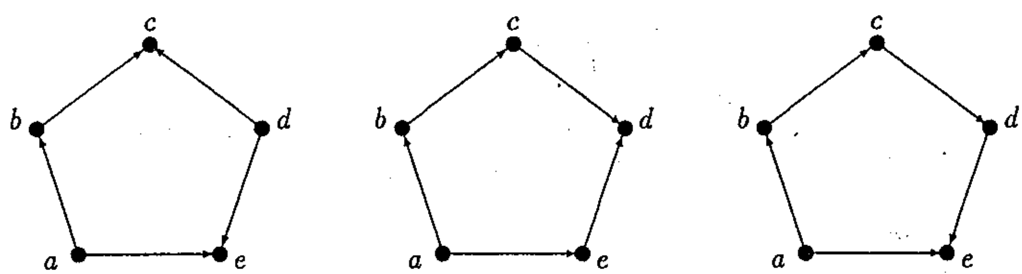
There are four orientations of the 4-cycle: the 4-circuit and the following



X Y Z

We represent X as $[a,b,c,d]_x$, Y as $[a,b,c,d]_y$, and Z as $[a,b,c,d]_z$. A $M(4,v)$ exists if and only if $v \equiv 0$ or 1 (mod 4), $v \neq 4$ [23]. A X−decomposition of D_v exists if and only if $v \equiv 0$ or 1 (mod 4), $v \neq 5$, a Y−decomposition of D_v exists if and only if $v \equiv 0$ or 1 (mod 4), $v \notin \{4,5\}$, and a Z−decomposition of D_v exists if and only if $v \equiv 1$ (mod 4) [10].

There are four orientations of the 5-cycle: the 5-circuit and the following



A_1 A_2 A_3

We represent A_i as $[a,b,c,d,e]_i$, for $i = 1,2,3$. A $M(5,v)$ exists if and only if $v \equiv 0$ or 1 (mod 5) [2]. This is also the spectrum of an A_i−decomposition of D_v for $i = 1,2,3$ [1].

A digraph d is said to be *self-converse* if reversing each of the arcs produces a digraph isomorphic to d. Each of the orientations of the 3-cycle, 4-cycle and 5-cycle are self-converse. These are the only cycles for which each orientation is self-converse [9]. Varma [26] gave necessary and sufficient conditions for the decomposition of D_v into self-converse orientations of 6-cycles, 7-cycles and 8-cycles.

An *automorphism* of a digraph decomposition of D_v is a permutation of the vertex set of D_v which fixes the collection of isomorphic digraphs in the decomposition. A decomposition of D_v admitting an automorphism consisting of a cycle of length v is said to be *cyclic*. A decomposition of D_v admitting an automorphism consisting of a fixed point and a cycle of length $v-1$ is said to be *rotational*. Necessary and sufficient conditions are known for the existence of cyclic $M(k,v)$s for $k = 3,4,5,6,7,8$ [6,15,16]. Necessary and sufficient conditions are known for the existence of rotational $M(k,v)$s for $k = 3,4,5$ [4,18]. Cyclic directed triple systems exist if and only if $v \equiv 1$, 4, or 7 (mod 12) [7] and rotational directed triple systems exist if and only if $v \equiv 0$ (mod 3) [5]. The purpose of this paper is to explore cyclic and rotational automorphisms of decompositions of D_v into orientations of 4-cycles and 5-cycles. In addition, we address the existence question for decompositions of D_v into copies of the non-self-converse orientations of the 6-cycle. Finally, we present some related results for cycle systems (i.e. decompositions of the complete graph into cycles of a given length).

2 Cyclic Decompositions

In this section, we give necessary and sufficient conditions for the existence of cyclic decompositions of D_v into orientations of 4-cycles and 5-cycles. Throughout this section, we assume D_v has vertex set $\mathbf{Z}_v$ and the automorphism is the permutation $\alpha = (0, 1, \ldots, v-1)$. With each arc (a, b) of D_v we associate a *difference* of $b - a$ (mod v). The existence of a cyclic X-decomposition and of a cyclic Y-decomposition of D_v implies a partitioning of the set of differences $\{1, 2, \ldots, v-1\}$ into difference 4-tuples (d_i, d_j, d_k, d_l) such that

$d_i + d_j + d_k \equiv d_l$ (mod v) for X-decompositions, or

$d_i + d_j \equiv d_k + d_l$ (mod v) for Y-decompositions.

Therefore, a necessary condition for either such decomposition is that $v - 1 \equiv 0$ (mod 4). We show this condition is sufficient in the next two theorems.

Theorem 2.1 *A cyclic X-decomposition of D_v exists if and only if $v \equiv 1$ (mod 4), $v \neq 5$.*

Proof. We consider two cases.

Case 1. Suppose $v \equiv 1$ (mod 8), say $v = 8t + 1$. Consider the blocks:

$[0, 1+i, 6t+2+2i, 2t+1+i]_x$ for $i = 0, 1, \ldots, t-1$, and

$[0, t+1+i, 1+2i, 5t+1+i]_x$ for $i = 0, 1, \ldots, t-1$.

Case 2. Suppose $v \equiv 5$ (mod 8), say $v = 8t + 5$, $t > 0$. Consider the blocks:

$[0, 1, 2t+3, 4t+6]_x$; $[0, 2+i, 1, 2t+5+2i]_x$ for $i = 0, 1, \ldots, t$, and

$[0, t+3+i, 1, 4t+8+2i]_x$ for $i = 0, 1, \ldots, t-2$ (omit if $t = 1$).

In both cases, the blocks, along with their images under α, form a cyclic X-decomposition of D_v. ∎

Theorem 2.2 *A cyclic Y-decomposition of D_v exists if and only if $v \equiv 1$ (mod 4), $v \neq 5$.*

Proof. Suppose $v = 4t + 1$, $t > 1$. Consider the blocks:

$[0, 2t-1, 2t-2, 4t-1]_y$, and $[0, 1+i, 4t-1, 2t+1+i]_y$ for $i = 0, 1, \ldots, t-2$.

These blocks, along with their images under α, form a cyclic Y-decomposition of D_v. ∎

Theorem 2.3 *A cyclic Z-decomposition of D_v exists if and only if $v \equiv 1$ (mod 4).*

Proof. The necessary conditions follow from the spectrum of a Z-decomposition of D_v. Suppose $v \equiv 1$ (mod 4), say $v = 4t + 1$. Consider the blocks:

$[0, 1+i, 2t+2+2i, 2t+1+i]_z$ for $i = 0, 1, \ldots, t-1$.

These blocks, along with their images under α, form a cyclic Z-decomposition of D_v. ∎

We now turn our attention to the existence of cyclic A_i–decompositions of D_v for $i = 1, 2, 3$. We have as a necessary condition:

Lemma 2.1 *If a cyclic A_i–decomposition of D_v exists, where $i = 1, 2, 3$, then $v \equiv 1, 11,$ or 16 (mod 20).*

Proof. The existence of such a decomposition implies a partitioning of the set of differences $\{1, 2, \ldots, v-1\}$ into difference 5-tuples $(d_i, d_j, d_k, d_l, d_m)$ such that

$d_i + d_j + d_k \equiv d_l + d_m \pmod v$ for A_1– and A_2–decompositions, or

$d_i + d_j + d_k + d_l \equiv d_m \pmod v$ for A_3–decompositions.

Therefore, $v - 1 \equiv 0 \pmod 5$ is necessary. Notice that from the above condition on the difference 5-tuples, we can partition the difference set into two sets A and B such that the sum of the elements of A is congruent to the sum of the elements of B modulo v. That is, there exists a and b such that $a + b = \frac{v(v-1)}{2}$ and $a \equiv b$ (mod v). But if $v \equiv 6 \pmod{20}$, say $v = 20t + 6$, then $a + b = (20t + 5)(10t + 3)$ which is odd. But if $a \equiv b \pmod{20t + 6}$ then $a + b$ is even. Therefore $v \equiv 6 \pmod{20}$ is not possible and the necessary conditions for such a system are $v \equiv 1$, 11, or 16 (mod 20). ∎

We now show the necessary conditions of Lemma 2.1 are also sufficient.

Theorem 2.4 *A cyclic A_1–decomposition of D_v exists if and only if $v \equiv 1, 11,$ or 16 (mod 5).*

Proof. We consider two cases.

Case 1. Suppose $v \equiv 1 \pmod{10}$, say $v = 10t + 1$. Consider the blocks:

$[0, 1 + i, 5t + 1, 4t - i, 4t - 1 - 2i]_1$ for $i = 0, 1, \ldots, t-1$, and

$[0, 2t + 2 + 2i, 7t + 3 + 3i, 8t + 4 + 4i, 6t + 2 + 2i]_1$ for $i = 0, 1, \ldots, t-1$.

Case 2. Suppose $v \equiv 16 \pmod{20}$, say $v = 20t + 16$. Consider the blocks:

$[0, 20t + 14 - 2i, 7t + 4 - i, 20t + 15, 1 + 2i]_1$ for $i = 0, 1, \ldots, 2t - 1$ (omit if $t = 0$),

$[0, 4t + 3, 13t + 9, 8t + 4, 4t + 1]_1$, $[0, 4t + 2, 19t + 14, 9t + 6, 5t + 4]_1$,

$[0, 10t + 9 + i, 3t + 3, 10t + 8 - i, 20t + 15 - 2i]_1$ for $i = 0, 1, \ldots, t$, and

$[0, 5t + 3 - i, 17t + 12 - 2i, 19t + 15, 14t + 12 + i]_1$ for $i = 0, 1, \ldots, t-1$ (omit if $t = 0$).

In both cases, the blocks, along with their images under α, form a cyclic A_1-decomposition of D_v. ∎

Theorem 2.5 *A cyclic A_2–decomposition of D_v exists if and only if $v \equiv 1, 11,$ or 16 (mod 5).*

Proof. We consider two cases.

Case 1. Suppose $v \equiv 1 \pmod{10}$, say $v = 10t + 1$. Consider the blocks:

$[0, 1 + i, 5t + 1, 5t - i, 4t - 1 - 2i]_2$ for $i = 0, 1, \ldots, t-1$, and

$[0, 2t + 2 + 2i, 7t + 3 + 3i, 5t + 1 + i, 6t + 2 + 2i]_2$ for $i = 0, 1, \ldots, t-1$.

Case 2. Suppose $v \equiv 16 \pmod{20}$, say $v = 20t + 16$. Consider the blocks:

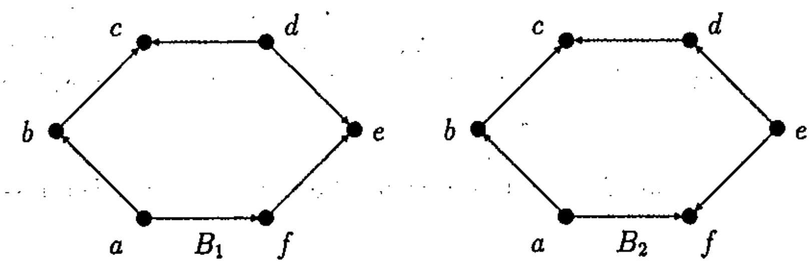
$[0, 20t + 14 - 2i, 7t + 4 - i, 7t + 6 + i, 7t + 5 - i]_2$ for $i = 0, 1, \ldots, 2t - 1$ (omit if $t = 0$), $[0, 4t + 3, 13t + 9, 9t + 6, 4t + 1]_2$, $[0, 16t + 14, 11t + 10, 15t + 12, 5t + 4]_2$,

$[0, 10t+9+i, 3t+3, 13t+10-i, 13t+11+i]_2$ for $i = 0, 1, \ldots, t$, and
$[0, 15t+13+i, 7t+6, 12t+9-i, 18t+13-2i]_2$ for $i = 0, 1, \ldots, t-1$ (omit if $t = 0$).
In both cases, the blocks, along with their images under α, form a cyclic A_2-decomposition of D_v. ∎

Theorem 2.6 *A cyclic A_3–decomposition of D_v exists if and only if $v \equiv 1, 11,$ or 16 (mod 5).*

Proof. We consider two cases.
Case 1. Suppose $v \equiv 1 \pmod{10}$, say $v = 10t+1$. Consider the blocks:
$[0, 1+i, t+2+2i, 5t+1, 5t-i]_3$ for $i = 0, 1, \ldots, t-1$, and
$[0, 2t+2+2i, t+1+i, 7t+3+3i, 5t+1+i]_3$ for $i = 0, 1, \ldots, t-1$.
Case 2. Suppose $v \equiv 16 \pmod{20}$, say $v = 20t+16$. Consider the blocks:
$[0, 1+2i, 20t+15, 7t+4-i, 7t+6+i]_3$ for $i = 0, 1, \ldots, 2t-1$ (omit if $t = 0$),
$[0, 4t+1, 8t+4, 13t+9, 9t+6]_3$, $[0, 5t+4, 9t+6, 19t+14, 15t+12]_3$,
$[0, 20t+15-2i, 10t+8-i, 3t+3, 13t+10-i]_3$ for $i = 0, 1, \ldots, t$, and
$[0, 14t+12+i, 19t+15, 17t+12-2i, 12t+9-i]_3$ for $i = 0, 1, \ldots, t-1$ (omit if $t = 0$).
In both cases, the blocks, along with their images under α, form a cyclic A_3-decomposition of D_v. ∎

We now consider the cyclic decomposition of D_v into the following orientations of the 6-cycle:



We represent B_1 as $[a, b, c, d, e, f]_1$ and B_2 as $[a, b, c, d, e, f]_2$. Neither of these orientations is self-converse. However, B_1 is the converse of B_2. Necessary and sufficient conditions are known for the decompositions of D_v into the remaining orientations of the 6-cycle (each of which is self-converse) [26]. Necessary conditions for the existence of B_1– or B_2–decompositions of D_v are that $v \equiv 0, 1,$ or $3 \pmod 6$.

Theorem 2.7 *A cyclic B_1–decomposition of D_v exists if and only if $v \equiv 1$ (mod 6).*

Proof. As above, a necessary condition for such a system is that the number of differences be divisible by 6. That is, $v \equiv 1 \pmod 6$ is necessary. We show sufficiency in three cases.
Case 1. Suppose $v \equiv 1 \pmod{12}$, say $v = 12t+1$. Consider the blocks:
$[0, 1+12i, 4+24i, 2+12i, 9+24i, 5+12i]_1$ for $i = 0, 1, \ldots, t-1$, and
$[0, 6+12i, 16+24i, 8+12i, 20+24i, 11+12i]_1$ for $i = 0, 1, \ldots, t-1$.
Case 2. Suppose $v \equiv 7 \pmod{24}$, say $v = 24t+7$. Consider the blocks:
$[0, 1+12i, 6+24i, 3+12i, 10+24i, 4+12i]_1$ for $i = 0, 1, \ldots, t-1$ (omit if $t = 0$),

$[0, 8+12i, 18+24i, 9+12i, 23+24i, 12+12i]_1$ for $i = 0, 1, \ldots, t-1$ (omit if $t = 0$),
$[0, 12t+5+12i, 24t+14+24i, 12t+7+12i, 24t+18+24i, 12t+8+12i]_1$ for $i = 0, 1, \ldots, t-1$ (omit if $t = 0$),
$[0, 12t+12+12i, 24t+26+24i, 12t+13+12i, 24t+31+24i, 12t+16+12i]_1$ for $i = 0, 1, \ldots, t-1$ (omit if $t = 0$), and $[0, 2, 12t+6, 3, 12t+4, 24t+5]_1$.

Case 3. Suppose $v \equiv 19 \pmod{24}$, say $v = 24t + 19$. Consider the blocks:
$[0, 1+12i, 6+24i, 3+12i, 10+24i, 4+12i]_1$ for $i = 0, 1, \ldots, t$,
$[0, 8+12i, 18+24i, 9+12i, 23+24i, 12+12i]_1$ for $i = 0, 1, \ldots, t-1$ (omit if $t = 0$),
$[0, 12t+12+12i, 24t+26+24i, 12t+13+12i, 24t+31+24i, 12t+16+12i]_1$ for $i = 0, 1, \ldots, t$,
$[0, 12t+17+12i, 24t+38+24i, 12t+19+12i, 24t+42+24i, 12t+20+12i]_1$ for $i = 0, 1, \ldots, t-1$ (omit if $t = 0$), and $[0, 2, 12t+11, 1, 12t+9, 24t+17]_1$.

In each case, the blocks, along with their images under α, form a cyclic B_1-decomposition of D_v. ∎

Since B_2 is the converse of B_1, the existence of a cyclic B_1-decomposition of D_v implies the existence of a cyclic B_2 decomposition of D_v (and conversely). We therefore have:

Theorem 2.8 *A cyclic B_2−decomposition of D_v exists if and only if $v \equiv 1 \pmod 6$.*

3 Rotational Decompositions

In this section, we give necessary and sufficient conditions for the existence of rotational decompositions of D_v into orientations of 4-cycles and 5-cycles. Throughout this section, we assume D_v has vertex set $\{\infty\} \bigcup \mathbb{Z}_{v-1}$ and the automorphism is the permutation $\beta = (\infty)(0, 1, \ldots, v-2)$.

Lemma 3.1 *If a rotational X− or Y−decomposition of D_v exists, then $v \equiv 0 \pmod 4$.*

Proof. The existence of a rotational X− or Y−decomposition of D_v implies the partitioning of the set of differences $\{1, 2, \ldots, v-2\} \setminus \{d_1, d_2\}$, where d_1 and d_2 are two differences, into difference 4-tuples (d_i, d_j, d_k, d_l) such that
$d_i + d_j + d_k \equiv d_l \pmod v$ for X−decompositions, or
$d_i + d_j \equiv d_k + d_l \pmod v$ for Y−decompositions.
Therefore, a necessary condition for either such decomposition is that $v - 4 \equiv 0 \pmod 4$. ∎

We show this condition is sufficient in the next two theorems.

Theorem 3.1 *A rotational X−decomposition of D_v exists if and only if $v \equiv 0 \pmod 4$.*

Proof. We consider two cases.
Case 1. Suppose $v \equiv 0 \pmod 8$, say $v = 8t$. Consider the blocks:
$[6t-1, \infty, 0, 6t-2]_x$, $[0, 1+i, 6t+2i, 2t+i]_x$ for $i = 0, 1, \ldots, t-1$, and

$[0, t+1+i, 8t+2i, 5t-1+i]_x$ for $i = 0, 1, \ldots, t-2$ (omit if $t = 1$).

Case 2. Suppose $v \equiv 4 \pmod 8$, say $v = 8t + 4$. Consider the blocks:

$[1, \infty, 0, 6t+2]_x$, $[0, 1+i, 6t+4+2i, 2t+1+i]_x$ for $i = 0, 1, \ldots, t-1$ (omit if $t = 0$), and

$[0, t+1+i, 8t+4+2i, 5t+1+i]_x$ for $i = 0, 1, \ldots, t-1$ (omit if $t = 0$).

In both cases, the blocks, along with their images under the permutation β, form a rotational X-decomposition of D_v. ∎

Theorem 3.2 *A rotational $Y-$decomposition of D_v exists if and only if $v \equiv 0 \pmod 4$, $v \neq 4$.*

Proof. Suppose $v \equiv 0 \pmod 4$, say $v = 4t$, $t \geq 2$. Consider the blocks:

$[1, \infty, 4t-3, 0]_y$, and $[0, 1+i, 4t-3, 2t-1+i]_y$ for $i = 0, 1, \ldots, t-2$.

These blocks along with their images under the permutation $\pi = (\infty)(0, 1, \ldots, 4t-2)$, form a Y-decomposition of D_v where the point set of D_v is $\{\infty\} \bigcup \mathbf{Z}_{4t-1}$. ∎

We note that Theorems 2.1, 2.2, 2.3, 3.1, and 3.2 combine to give direct constructions of $X-$, $Y-$, and $Z-$decompositions of D_v for all admissible v.

Concerning rotational $Z-$decompositions of D_v we have:

Theorem 3.3 *A rotational $Z-$decomposition of D_v does not exist for any v.*

Proof. Such a system must have a block containing the fixed point ∞. So there must be either a block of the form $A = [a, \infty, b, c]_z$ or $B = [c, a, \infty, b]_z$. If we apply β^{b-a} to block A, we get the arc (a, ∞) twice in the decomposition, a contradiction. If we apply β^{b-a} to block B, we get the arc (∞, a) twice in the decomposition, another contradiction. Therefore, a rotational $Z-$decomposition of D_v does not exist. ∎

We now turn our attention to rotational A_i-decompositions of D_v for $i = 1, 2, 3$.

Lemma 3.2 *If a rotational A_i-decomposition of D_v exists for $i = 1, 2$ or 3, then $v \equiv 0 \pmod 5$.*

Proof. As in Lemma 3.1, the existence of such a decomposition implies a partitioning of the set $\{1, 2, \ldots, v-2\} \setminus \{d_1, d_2, d_3\}$ into difference 5-tuples such that

$d_i + d_j + d_k \equiv d_l + d_m \pmod{v-1}$ for $i = 1, 2$, or

$d_i + d_j + d_k + d_l \equiv d_m \pmod{v-1}$ for $i = 3$.

Therefore, a necessary condition for such a system is that $v - 5 \equiv 0 \pmod 5$. ∎

We show this condition is sufficient in the next three theorems.

Theorem 3.4 *A rotational A_1-decomposition of D_v exists if and only if $v \equiv 0 \pmod 5$.*

Proof. We consider two cases.

Case 1. Suppose $v \equiv 0 \pmod{10}$, say $v = 10t$. Consider the blocks:

$[0, 1+i, 4t+1+2i, 7t+1+3i, 7t+2i]_1$ for $i = 0, 1, \ldots, t-1$,

$[0, t+2+2i, 7t+i, 4t-1, 3t-3-2i]_1$ for $i = 0, 1, \ldots, t-2$ (omit if $t = 1$), and
$[0, \infty, 6t, 1, 3t]_1$.

Case 2. If $v = 5$, consider the block $[0, \infty, 2, 3, 1]_1$. Next, suppose $v \equiv 5 \pmod{10}$, say $v = 10t + 5$, $t \geq 1$. Consider the blocks:
$[0, 6t+3-i, 9t+4, 6t+4+i, 1+2i]_1$ for $i = 0, 1, \ldots, t-1$,
$[0, 4+2i, 7t+7+i, 3, 10t+3-2i]_1$ for $i = 0, 1, \ldots, t-2$ (omit if $t = 1$),
$[0, 6t+4, t+1, 3t+2, 8t+5]_1$, and $[0, \infty, 5t+2, 5t, 10t+2]_1$.
These blocks, along with their images under β, form a rotational A_1-decomposition of D_v. ∎

Theorem 3.5 *A rotational A_2–decomposition of D_v exists if and only if $v \equiv 0 \pmod 5$.*

Proof. We consider two cases.
Case 1. Suppose $v \equiv 0 \pmod{10}$, say $v = 10t$. Consider the blocks:
$[0, 1+i, 4t+1+2i, 4t+i, 7t+2i]_2$ for $i = 0, 1, \ldots, t-1$,
$[0, t+2+2i, 7t+i, 6t-2-i, 3t-3-2i]_2$ for $i = 0, 1, \ldots, t-2$ (omit if $t = 1$), and
$[0, 3t-1, \infty, 9t-1, 3t]_2$.
Case 2. Suppose $v \equiv 5 \pmod{10}$, say $v = 10t + 5$. Consider the blocks:
$[0, 3t+1+i, 9t+4, 3t+1+i, 3t-i]_2$ for $i = 0, 1, \ldots, t-1$,
$[0, 7t+3-i, 7t+7+i, 7t+3-i, 10t+3-2i]_2$ for $i = 0, 1, \ldots, t-2$ (omit if $t = 1$),
$[0, 6t+4, t+1, 6t+4, 8t+5]_2$, and $[0, \infty, 5t-2, 5t, 10t+2]_2$.
These blocks, along with their images under β, form a rotational A_2-decomposition of D_v. ∎

Theorem 3.6 *A rotational A_3–decomposition of D_v exists if and only if $v \equiv 0 \pmod 5$.*

Proof. We consider two cases.
Case 1. Suppose $v \equiv 0 \pmod{10}$, say $v = 10t$. Consider the blocks:
$[0, 6t-1, \infty, 1, 3t]_3$, $[0, 1+i, 7t, 4t+1+2i, 4t+i]_3$ for $i = 0, 1, \ldots, t-1$, and
$[0, t+2+2i, 4t+3+3i, 7t+i, 6t-2-i]_3$ for $i = 0, 1, \ldots, t-2$ (omit if $t = 1$).
Case 2. If $v = 5$, consider the block $[0, 1, \infty, 3, 2]_3$. Next, suppose $v \equiv 5 \pmod{10}$, say $v = 10t + 5$, $t \geq 1$. Consider the blocks:
$[0, 1+2i, 6t+4+i, 9t+4, 3t+1+i]_3$ for $i = 0, 1, \ldots, t-1$,
$[0, 10t+3-2i, 10t+7, 7t+7+i, 7t+3-i]_3$ for $i = 0, 1, \ldots, t-2$ (omit if $t = 1$),
$[0, 8t+3, 3t, 8t+3, 6t+4]_3$, and $[0, \infty, 5t-2, 5t, 10t+2]_3$.
These blocks, along with their images under β, form a rotational A_3-decomposition of D_v. ∎

We note that Theorems 2.4, 2.5, 2.6, 3.4, 3.5 and 3.6 combine to give direct constructions of A_i–decompositions of D_v for all admissible v except for $v \equiv 6 \pmod{20}$.

We now consider B_1– and B_2–decompositions of D_v which admit a rotational automorphism. We have:

Theorem 3.7 *A rotational B_1–decomposition of D_v exists if and only if $v \equiv 0$ (mod 6).*

Proof. An argument similar to those in Lemmas 3.1 and 3.2 shows that $v \equiv 0$ (mod 6) is necessary. For sufficiency, we consider two cases.
Case 1. Suppose $v \equiv 0$ (mod 12), say $v = 12t$. Consider the blocks:
$[0, 1+12i, 4+24i, 2+12i, 9+24i, 5+12i]_1$ for $i = 0, 1, \ldots, t-1$,
$[0, 6+12i, 16+24i, 8+12i, 20+24i, 11+12i]_1$ for $i = 0, 1, \ldots, t-2$ (omit if $t = 1$),
and $[0, \infty, 12t-8, 12t-3, 12t-5, 12t-2]_1$.
Case 2. If $v = 6$, consider the block $[0, \infty, 3, 1, 2, 4]_1$. Next suppose $v \equiv 6$ (mod 12), say $v = 12t+6$, $t \geq 1$. Consider the blocks:
$[0, 1+12i, 4+24i, 2+12i, 9+24i, 5+12i]_1$ for $i = 0, 1, \ldots, t-1$,
$[0, 6+12i, 16+24i, 8+12i, 20+24i, 11+12i]_1$ for $i = 0, 1, \ldots, t-1$, and
$[0, \infty, 12t-3, 12t, 12t-1, 12t+1]_1$.
In both cases, the blocks, along with their images under β, form a rotational B_1-decomposition of D_v. ∎

Since B_2 is the converse of B_1, the existence of a rotational B_1–decomposition of D_v implies the existence of a rotational B_2–decomposition of D_v (and conversely). We therefore have:

Theorem 3.8 *A rotational B_2–decomposition of D_v exists if and only if $v \equiv 0$ (mod 6).*

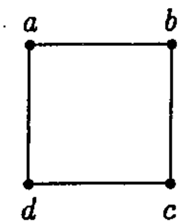
We note that Theorems 2.7, 2.8, 3.7, and 3.8 combine to give the existence of a B_1- or B_2-decomposition of D_v when $v \equiv 0$ or 1 (mod 6). We leave the case $v \equiv 3$ (mod 6) open.

4 Related Results For Cycle Systems

The decomposition of a graph is defined similarly to the decomposition of a digraph, and an automorphism of a graph decomposition is analogous to an automorphism of a digraph decomposition. An n–*cycle system* of order v, denoted $nCS(v)$, is a decomposition of K_v (the complete graph on v vertices) into cycles of length n. A $4CS(v)$ exists if and only if $v \equiv 1$ (mod 8) [12], a $6CS(v)$ exists if and only if $v \equiv 1$ or 19 (mod 12) [20], and an $8CS(v)$ exists if and only if $v \equiv 1$ (mod 16) [12]. For a survey of results on cycle systems see [13]. A cyclic 3-cycle system (or Steiner triple system) exists if and only if $v \equiv 1$ or 3 (mod 6), $v \neq 9$ [17]. Cyclic cycle systems in general are explored in [20] and [21]. We slightly generalize the idea of a rotational automorphism by defining a k–*rotational* automorphism as one consisting of a fixed point and k cycles each of length $\frac{v-1}{k}$. k–rotational $3CS(v)$s are explored in [3, 19] in which the spectrum is determined for $k = 1, 2, 3, 4, 6$. An n–cycle system is said to be *reverse* if it admits an automorphism consisting of a fixed point and $\frac{v-1}{2}$ transpositions. A reverse $3CS(v)$ exists if and only if $v \equiv 1$, 3, 9 or 19 (mod 24) [8, 22, 24, 25]. In this section, we give necessary and sufficient conditions for the existence of k–rotational $nCS(v)$s and reverse $nCS(v)$s for all k and for $n = 4, 6, 8$.

We consider k-rotational $nCS(v)$s on the point set $\{\infty\} \bigcup Z_N \times Z_k$, where $N = \frac{v-1}{k}$, and with automorphism $\pi_k = (\infty)(0_0, 1_0, \ldots, (N-1)_0) \cdots (0_{k-1}, 1_{k-1}, \ldots, (N-1)_{k-1})$, where we represent the ordered pair $(x, y) \in Z_N \times Z_k$ as x_y.

We will represent the 4-cycle



by any cyclic shift of (a, b, c, d) or (d, c, b, a). We have the following necessary condition:

Lemma 4.1 *If a k-rotational $4CS(v)$ exists, then k is even.*

Proof. Suppose there is a k-rotational $4CS(v)$ with point set and automorphism as described above. A set of $n-$cycles is said to be a *set of base $n-$cycles* for an $nCS(v)$ under the automorphism π if the images of the $n-$cycles under the powers of π produce the $nCS(v)$. For each $i \in Z_k$, there must be exactly one 4-cycle in a set of base 4-cycles under the automorphism π_k which contains the edge (∞, a_i), for some $a \in Z_N$. Base 4-cycles containing ∞ must be of one of the following types:

1. (∞, x_i, y_i, z_j) where $i \neq j$, or
2. (∞, x_i, y_j, z_m) where i, j and m are distinct.

Each of these types of 4-cycles contains exactly two edges of the form (∞, a_i) where $a_i \in Z_N \times Z_k$. Therefore, k must be even. ∎

Notice that the argument of Lemma 4.1 can be extended to show that if a k-rotational $nCS(v)$ exists where n is even, then k must be even.

We can now establish sufficiency.

Lemma 4.2 *A 2-rotational $4CS(v)$ exists if and only if $v \equiv 1 \pmod 8$.*

Proof. Suppose $v \equiv 1 \pmod 8$, say $v = 8t + 1$. Let $N = 4t$. Consider the collection of 4-cycles:

$(\infty, 0_0, t_1, (2t)_1)$, $(0_0, t_0, (2t)_0, (3t)_0)$, $(0_0, 0_1, (2t)_0, (2t)_1)$, $(0_0, (3t)_1, t_1, (2t)_0)$,
$(0_0, i_1, (2i)_1, (2t + i)_0)$ for $i = 1, 2, \ldots, t - 1$ (omit if $t = 1$), and
$(0_0, (4t - i)_1, (2t)_1, i_0)$ for $i = 1, 2, \ldots, t - 1$ (omit if $t = 1$).

These blocks, along with their images under π_2, form a 2-rotational $4CS(v)$. ∎

The results of Lemmas 4.1 and 4.2 allow us to establish necessary and sufficient conditions for the existence of a k-rotational $4CS(v)$, for all k.

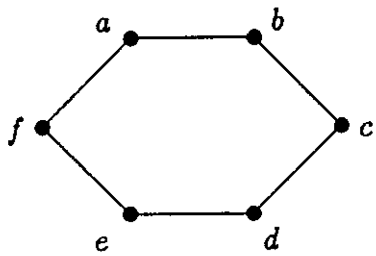
Theorem 4.1 *A k-rotational $4CS(v)$ exists if and only if $v \equiv 1 \pmod 8$, k is even and $v \equiv 1 \pmod k$.*

Proof. In light of Lemma 4.1, the necessary conditions follow trivially. Now suppose v and k satisfy the stated hypotheses. Then there is a 2-rotational $4CS(v)$ admitting automorphism π_2. Since $v \equiv 1 \pmod{k}$, we have $\pi_2^{k/2} = \pi_k$ and so the 2-rotational $4CS(v)$ is also k-rotational. ∎

Theorem 4.1 immediately classifies reverse $4CS(v)$s:

Corollary 4.1 *A reverse $4CS(v)$ exists if and only if $v \equiv 1 \pmod{8}$.*

We will represent the 6-cycle



by any cyclic shift of (a, b, c, d, e, f) or (f, e, d, c, b, a). We now consider k-rotational $6CS(v)$s. As with Lemma 4.1, a necessary condition for the existence of a k–rotational $6CS(v)$ is that k is even.

We now establish sufficiency. In each of the constructions, we represent certain blocks with the following notation:

$A_j(a,i) = (0_j, (a+6+12i)_j, (2a+8+24i)_j, (3a+8+36i)_j, (2a+4+24i)_j, (a+1+12i)_j)$,
$B_j(b,i) = (0_j, (b+6+12i)_j, (2b+10+24i)_j, (3b+10+36i)_j, (2b+5+24i)_j, (b+2+12i)_j)$,
$C(c,i) = (0_0, (c+6+12i)_1, 2_0, (c+4+12i)_1, 1_0, (c+1+12i)_1)$, and
$D(d,i) = (0_0, (d+6+12i)_1, 1_0, (d+5+12i)_1, 2_0, (d+2+12i)_1)$.

Lemma 4.3 *If $v \equiv 1 \pmod{48}$ then there exists a 2-rotational $6CS(v)$.*

Proof. First, suppose that $v = 49$. Consider the collection of 6-cycles:

$(\infty, 3_0, 0_0, 2_1, 5_1, 11_1)$, $(0_0, 6_0, 10_1, 22_1, 18_0, 12_0)$, $(0_0, 11_1, 1_0, 8_1, 3_0, 9_1)$,
$(0_0, 8_1, 8_0, 16_1, 16_0, 0_1)$, $(0_0, 1_1, 2_1, 4_1, 1_0, 2_0)$, $C(12,0)$, $D(17,0)$,
$(0_j, 4_j, 8_j, 12_j, 16_j, 20_j)$ for $j = 0, 1$, and $B_j(5,0)$ for $j = 0, 1$.

Now suppose $v \equiv 1 \pmod{48}$, say $v = 48t + 1$ where $t > 1$. Consider the collection of 6-cycles:

$(\infty, 3_0, 0_0, 2_1, 5_1, 9_1)$, $(0_0, 4_0, 8_1, (12t+8)_1, (12t+4)_0, (12t)_0)$,
$(0_0, (8t)_1, (8t)_0, (16t)_1, (16t)_0, 0_1)$, $(0_0, 1_1, 2_1, 4_1, 1_0, 2_0)$,
$(0_j, (4t)_j, (8t)_j, (12t)_j, (16t)_j, (20t)_j)$ for $j = 0, 1$,

along with

Case 1. if $t \equiv 0 \pmod{3}$, the 6-cycles

$A_j(5,i)$ for $i = 0, 1, \ldots, \frac{t-3}{3}$ for $j = 0, 1$,
$B_j(10,i)$ for $i = 0, 1, \ldots, \frac{t-6}{3}$ for $j = 0, 1$ (omit if $t = 3$),
$(0_j, (4t+4)_j, (8t+6)_j, (12t+7)_j, (8t+1)_j, (4t-2)_j)$ for $j = 0, 1$,
$B_j(4t+5,i)$ for $i = 0, 1, \ldots, \frac{2t-3}{3}$ for $j = 0, 1$,
$A_j(4t+12,i)$ for $i = 0, 1, \ldots, \frac{2t-6}{3}$ for $j = 0, 1$,
$C(5,i)$ for $i = 0, 1, \ldots, \frac{2t-3}{3}$, $D(10,i)$ for $i = 0, 1, \ldots, \frac{2t-6}{3}$,
$(0_0, (8t+6)_1, 2_0, (8t+5)_1, 4_0, (8t+2)_1)$, $D(8t+5,i)$ for $i = 0, 1, \ldots, \frac{4t-3}{3}$, and

$C(8t+12,i)$ for $i=0,1,\ldots,\frac{4t-6}{3}$;

Case 2. if $t \equiv 1 \pmod 3$, $t>1$, the 6-cycles

$A_j(5,i)$ for $i=0,1,\ldots,\frac{t-4}{3}$ for $j=0,1$,

$B_j(10,i)$ for $i=0,1,\ldots,\frac{t-7}{3}$ for $j=0,1$ (omit if $t=4$),

$(0_j,(4t+2)_j,(8t)_j,(12t-6)_j,(8t-7)_j,(4t-4)_j)$ for $j=0,1$,

$(0_j,(4t+7)_j,(8t+13)_j,(12t+12)_j,(8t+7)_j,(4t+3)_j)$ for $j=0,1$,

$A_j(4t+8,i)$ for $i=0,1,\ldots,\frac{2t-5}{3}$ for $j=0,1$,

$B_j(4t+13,i)$ for $i=0,1,\ldots,\frac{2t-5}{3}$ for $j=0,1$,

$C(5,i)$ for $i=0,1,\ldots,\frac{2t-5}{3}$, $D(10,i)$ for $i=0,1,\ldots,\frac{2t-5}{3}$,

$(0_0,(8t+3)_1,1_0,(8t)_1,3_0,(8t+1)_1)$, $C(8t+4,i)$ for $i=0,1,\ldots,\frac{4t-4}{3}$, and

$D(8t+9,i)$ for $i=0,1,\ldots,\frac{4t-4}{3}$;

Case 3. if $t \equiv 2 \pmod 3$, the 6-cycles

$A_j(5,i)$ for $i=0,1,\ldots,\frac{t-5}{3}$ for $j=0,1$ (omit if $t=2$),

$B_j(10,i)$ for $i=0,1,\ldots,\frac{t-5}{3}$ for $j=0,1$ (omit if $t=2$),

$(0_j,(4t+3)_j,(8t+2)_j,(12t)_j,(8t-2)_j,(4t-3)_j)$ for $j=0,1$,

$A_j(4t+4,i)$ for $i=0,1,\ldots,\frac{2t-4}{3}$ for $j=0,1$,

$B_j(4t+9,i)$ for $i=0,1,\ldots,\frac{2t-4}{3}$ for $j=0,1$,

$C(5,i)$ for $i=0,1,\ldots,\frac{2t-4}{3}$, $D(10,i)$ for $i=0,1,\ldots,\frac{2t-7}{3}$ (omit if $t=2$),

$(0_0,(8t+2)_1,3_0,(8t)_1,2_0,(8t-4)_1)$, $D(8t+1,i)$ for $i=0,1,\ldots,\frac{4t-2}{3}$, and

$C(8t+8,i)$ for $i=0,1,\ldots,\frac{4t-5}{3}$.

In each case, the blocks, along with their images under π_2, form a 2-rotational $6CS(v)$. ∎

Lemma 4.4 *If $v \equiv 9 \pmod{24}$ then there exists a 2-rotational $6CS(v)$.*

Proof. First, suppose that $v=9$. Consider the collection of 6-cycles:

$(\infty,0_0,0_1,1_0,3_1,2_1)$ and $(0_0,1_0,2_1,0_1,3_0,2_0)$.

Now, suppose $v \equiv 9 \pmod{24}$, say $v=24t+9$ where $t>0$. Consider the collection of 6-cycles:

$(\infty,2_0,0_0,3_1,4_1,8_1)$, $(0_0,3_0,4_0,6_1,3_1,1_1)$, $(0_0,9_1,15_1,7_0,17_1,6_0)$,

$(0_0,4_0,4_1,(6t+6)_1,(6t+6)_0,(6t+2)_0)$, $(0_0,5_1,10_1,6_0,12_1,5_0)$,

$C(16,i)$ for $i=0,1,\ldots,t-2$ (omit if $t=1$),

$D(21,i)$ for $i=0,@1,\ldots,t-2$ (omit if $t=1$),

along with

Case 1. if $t \equiv 1 \pmod 2$, the 6-cycles

$(0_0,13_1,20_1,8_0,22_1,7_0)$, $A_j(8,i)$ for $i=0,1,\ldots,\frac{t-3}{2}$ for $j=0,1$ (omit if $t=1$), and

$B_j(13,i)$ for $i=0,1,\ldots,\frac{t-3}{2}$ for $j=0,1$ (omit if $t=1$);

Case 2. if $t \equiv 0 \pmod 2$, $t>0$, the 6-cycles

$(0_0,12_1,20_1,7_0,22_1,8_0)$, $B_j(7,i)$ for $i=0,1,\ldots,\frac{t-2}{2}$ for $j=0,1$, and

$A_j(14,i)$ for $i=0,1,\ldots,\frac{t-4}{2}$ for $j=0,1$ (omit if $t=2$).

In each case, the blocks, along with their images under π_2, form a 2-rotational $6CS(v)$. ∎

Lemma 4.5 *If $v \equiv 13 \pmod{48}$ then there exists a 2-rotational $6CS(v)$.*

Proof. First, suppose that $v=13$. Consider the collection of blocks:

$(\infty, 0_0, 2_0, 0_1, 1_1, 3_1)$, $(0_0, 1_0, 4_1, 1_1, 4_0, 3_0)$, $(0_0, 1_1, 2_0, 3_1, 4_0, 5_1)$, and $(0_0, 2_1, 2_0, 4_1, 4_0, 0_1)$.

Next, suppose that $v = 61$. Consider the collection of blocks:

$(\infty, 3_0, 0_0, 2_1, 5_1, 9_1)$, $(0_0, 4_0, 21_1, 36_1, 19_0, 15_0)$, $(0_0, 5_1, 11_1, 4_0, 10_1, 6_0)$, $(0_0, 1_1, 2_1, 4_1, 1_0, 2_0)$, $(0_0, 10_1, 10_0, 20_1, 20_0, 0_1)$, $(0_0, 9_1, 16_1, 4_0, 15_1, 7_0)$, $(0_j, 5_j, 10_j, 15_j, 20_j, 25_j)$ for $j = 0, 1$, $B_j(8, 0)$ for $j = 0, 1$, $(0_0, 14_1, 23_1, 7_0, 22_1, 9_0)$, $C(18, 0)$ for $j = 0, 1$, and $D(23, 0)$ for $j = 0, 1$.

Now, suppose $v \equiv 13 \pmod{48}$, say $v = 48t + 13$ where $t > 1$. Consider the collection of 6-cycles:

$(\infty, 3_0, 0_0, 2_1, 5_1, 9_1)$, $(0_0, 4_0, 21_1, (12t+24)_1, (12t+7)_0, (12t+3)_0)$, $(0_0, 1_1, 2_1, 4_1, 1_0, 2_0)$, $(0_0, (8t+2)_1, (8t+2)_0, (16t+4)_1, (16t+4)_0, 0_1)$, $(0_j, (4t+1)_j, (8t+2)_j, (12t+3)_j, (16t+4)_j, (20t+5)_j)$ for $j = 0, 1$, $(0_0, 5_1, 10_1, 6_0, 12_1, 5_0)$, $(0_0, 9_1, 15_1, 7_0, 17_1, 6_0)$, $(0_0, 13_1, 20_1, 8_0, 22_1, 7_0)$,

along with

Case 1. if $t \equiv 0 \pmod 3$, $t > 0$, the 6-cycles

$A_j(8, i)$ for $i = 0, 1, \ldots, \frac{t-3}{3}$ for $j = 0, 1$,
$B_j(13, i)$ for $i = 0, 1, \ldots, \frac{t-6}{3}$ for $j = 0, 1$ (omit if $t = 3$),
$A_j(4t+3, i)$ for $i = 0, 1, \ldots, \frac{2t-3}{3}$ for $j = 0, 1$,
$B_j(4t+8, i)$ for $i = 0, 1, \ldots, \frac{2t-3}{3}$ for $j = 0, 1$, $D(16, i)$ for $i = 0, 1, \ldots, \frac{2t-6}{3}$,
$C(23, i)$ for $i = 0, 1, \ldots, \frac{2t-9}{3}$ (omit if $t = 3$), $(0_0, (8t+5)_1, 1_0, (8t+2)_1, 3_0, (8t+3)_1)$,
$C(8t+6, i)$ for $i = 0, 1, \ldots, \frac{4t-3}{3}$, and $D(8t+11, i)$ for $i = 0, 1, \ldots, \frac{4t-3}{3}$;

Case 2. if $t \equiv 1 \pmod 3$, $t > 1$, the 6-cycles

$A_j(8, i)$ for $i = 0, 1, \ldots, \frac{t-4}{3}$ for $j = 0, 1$,
$B_j(13, i)$ for $i = 0, 1, \ldots, \frac{t-7}{3}$ for $j = 0, 1$ (omit if $t = 4$),
$(0_j, (4t+7)_j, (8t+7)_j, (12t+4)_j, (8t+1)_j, (4t-1)_j)$ for $j = 0, 1$,
$(0_j, (4t+10)_j, (8t+16)_j, (12t+21)_j, (8t+12)_j, (4t+4)_j)$ for $j = 0, 1$,
$A_j(4t+11, i)$ for $i = 0, 1, \ldots, \frac{2t-5}{3}$ for $j = 0, 1$,
$B_j(4t+16, i)$ for $i = 0, 1, \ldots, \frac{2t-5}{3}$ for $j = 0, 1$,
$D(16, i)$ for $i = 0, 1, \ldots, \frac{2t-8}{3}$, $C(23, i)$ for $i = 0, 1, \ldots, \frac{2t-8}{3}$,
$(0_0, (8t+4)_1, 3_0, (8t+2)_1, 2_0, (8t-2)_1)$, $D(8t+3, i)$ for $i = 0, 1, \ldots, \frac{4t-1}{3}$, and
$C(8t+10, i)$ for $i = 0, 1, \ldots, \frac{4t-4}{3}$;

Case 3. if $t \equiv 2 \pmod 3$, the 6-cycles

$A_j(8, i)$ for $i = 0, 1, \ldots, \frac{t-5}{3}$ for $j = 0, 1$ (omit if $t = 2$),
$B_j(13, i)$ for $i = 0, 1, \ldots, \frac{t-5}{3}$ for $j = 0, 1$ (omit if $t = 2$),
$B_j(4t, i)$ for $i = 0, 1, \ldots, \frac{2t-1}{3}$ for $j = 0, 1$,
$A_j(4t+7, i)$ for $i = 0, 1, \ldots, \frac{2t-4}{3}$ for $j = 0, 1$,
$D(16, i)$ for $i = 0, 1, \ldots, \frac{2t-7}{3}$ (omit if $t = 2$),
$C(23, i)$ for $i = 0, 1, \ldots, \frac{2t-7}{3}$ (omit if $t = 2$), $(0_0, (8t+8)_1, 2_0, (8t+7)_1, 3_0, (8t+3)_1)$,
$D(8t+7, i)$ for $i = 0, 1, \ldots, \frac{4t-2}{3}$, and $C(8t+14, i)$ for $i = 0, 1, \ldots, \frac{4t-5}{3}$.

In each case, the blocks, along with their images under π_2, form a 2-rotational $6CS(v)$. ∎

Lemma 4.6 *If $v \equiv 21 \pmod{24}$ then there exists a 2-rotational $6CS(v)$.*

Proof. Suppose $v \equiv 21 \pmod{24}$, say $v = 24k + 21$. Consider the collection of 6-cycles:

$(0_0, 3_0, 2_0, 6_1, 4_1, 1_1)$, $D(3, i)$ for $i = 0, 1, \ldots, t$,

$C(10,i)$ for $i=0,1,\ldots,t-1$ (omit if $t=0$),
along with
Case 1. if $t\equiv 0 \pmod 2$, the 6-cycles
$(\infty, 2_0, 0_0, 2_1, 3_1, 7_1)$, $(0_0, 4_0, 4_1, (6t+9)_1, (6t+9)_0, (6t+5)_0)$,
$A_j(5,i)$ for $i=0,1,\ldots,\frac{t-2}{2}$ for $j=0,1$ (omit if $t=0$), and
$B_j(10,i)$ for $i=0,1,\ldots,\frac{t-2}{2}$ for $j=0,1$ (omit of $t=0$);
Case 2. if $t\equiv 1 \pmod 2$, the 6-cycles
$(\infty, 2_0, 0_0, 2_1, 3_1, 8_1)$, $(0_0, 5_0, 5_1, (6t+10)_1, (6t+10)_0, (6t+5)_0)$,
$(0_j, 4_j, 12_j, 22_j, 16_j, 9_j)$ for $j=0,1$,
$A_j(11,i)$ for $i=0,1,\ldots,\frac{t-3}{2}$ for $j=0,1$ (omit if $t=1$), and
$B_j(16,i)$ for $i=0,1,\ldots,\frac{t-3}{2}$ for $j=0,1$ (omit if $t=1$).
In each case, the blocks, along with their images under π_2, form a 2-rotational $6CS(v)$. ∎

Lemma 4.7 *If $v\equiv 25 \pmod{48}$ then there exists a 2-rotational $6CS(v)$.*

Proof. First suppose that $v=25$. Consider the collection of blocks:
$(\infty, 0_0, 5_0, 11_1, 2_1, 7_1)$, $(0_0, 3_0, 5_1, 11_1, 9_0, 6_0)$, $(0_0, 1_1, 2_1, 6_1, 3_0, 4_0)$,
$(0_0, 4_1, 4_0, 8_1, 8_0, 0_1)$, $D(5,0)$, $(0_j, 2_j, 4_j, 6_j, 8_j, 10_j)$ for $j=0,1$.
Now suppose $v\equiv 25 \pmod{48}$, say $v=48t+25$ where $t>0$. Consider the collection of 6-cycles:
$(\infty, 3_0, 0_0, 2_1, 5_1, 10_1)$, $(0_0, 5_0, 9_1, (12t+15)_1, (12t+11)_0, (12t+6)_0)$,
$(0_0, 1_1, 2_1, 4_1, 1_0, 2_0)$, $(0_0, (8t+4)_1, (8t+4)_0, (16t+8)_1, (16t+8)_0, 0_1)$,
$(0_j, (4t+2)_j, (8t+4)_j, (12t+6)_j, (16t+8)_j, (20t+10)_j)$ for $j=0,1$,
along with
Case 1. if $t\equiv 0 \pmod 3$, $t>0$, the 6-cycles
$B_j(4,i)$ for $i=0,1,\ldots,\frac{t-3}{3}$ for $j=0,1$,
$A_j(11,i)$ for $i=0,1,\ldots,\frac{t-6}{3}$ for $j=0,1$ (omit if $t=3$),
$(0_j, (4t+5)_j, (8t+6)_j, (12t+6)_j, (8t+2)_j, (4t-1)_j)$ for $j=0,1$,
$A_j(4t+6,i)$ for $i=0,1,\ldots,\frac{2t-3}{3}$ for $j=0,1$,
$B_j(4t+11,i)$ for $i=0,1,\ldots,\frac{2t-3}{3}$ for $j=0,1$, $C(5,i)$ for $i=0,1,\ldots,\frac{2t-3}{3}$,
$D(10,i)$ for $i=0,1,\ldots,\frac{2t-6}{3}$, $(0_0, (8t+6)_1, 3_0, (8t+4)_1, 2_0, (8t)_1)$,
$D(8t+5,i)$ for $i=0,1,\ldots,\frac{4t}{3}$, and $C(8t+12,i)$ for $i=0,1,\ldots,\frac{4t-3}{3}$;
Case 2. if $t\equiv 1 \pmod 3$, the 6-cycles
$B_j(4,i)$ for $i=0,1,\ldots,\frac{t-4}{3}$ for $j=0,1$ (omit if $t=1$),
$A_j(11,i)$ for $i=0,1,\ldots,\frac{t-4}{3}$ for $j=0,1$ (omit if $t=1$),
$(0_j, (4t+8)_j, (8t+13)_j, (12t+13)_j, (8t+7)_j, (4t+3)_j)$ for $j=0,1$,
$B_j(4t+7,i)$ for $i=0,1,\ldots,\frac{2t-2}{3}$ for $j=0,1$,
$A_j(4t+14,i)$ for $i=0,1,\ldots,\frac{2t-5}{3}$ for $j=0,1$ (omit if $t=1$),
$C(5,i)$ for $i=0,1,\ldots,\frac{2t-2}{3}$, $D(10,i)$ for $i=0,1,\ldots,\frac{2t-5}{3}$ (omit if $t=1$),
$(0_0, (8t+10)_1, 2_0, (8t+9)_1, 3_0, (8t+5)_1)$, $D(8t+9,i)$ for $i=0,1,\ldots,\frac{4t-1}{3}$, and
$C(8t+16,i)$ for $i=0,1,\ldots,\frac{4t-4}{3}$;
Case 3. if $t\equiv 2 \pmod 3$, the 6-cycles
$B_j(4,i)$ for $i=0,1,\ldots,\frac{t-5}{3}$ for $j=0,1$ (omit if $t=2$),
$A_j(11,i)$ for $i=0,1,\ldots,\frac{t-5}{3}$ for $j=0,1$ (omit if $t=2$),
$(0_j, (4t+4)_j, (8t+3)_j, (12t-1)_j, (8t-2)_j, (4t-2)_j)$ for $j=0,1$,
$B_j(4t+3,i)$ for $i=0,1,\ldots,\frac{2t-1}{3}$ for $j=0,1$,

In each case, the blocks, along with their images under π_2, form a 2-rotational $6CS(v)$. ∎

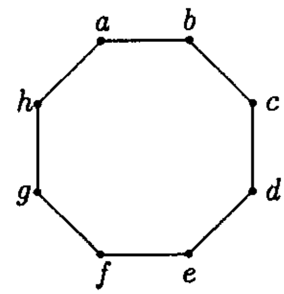
As in Theorem 4.1, the results of Lemmas 4.3 to 4.8 allow us to establish necessary and sufficient conditions for the existence of a k-rotational $6CS(v)$, for all k.

Theorem 4.2 *A k-rotational $6CS(v)$ exists if and only if $v \equiv 1$ or 9 (mod 12), k is even and $v \equiv 1$ (mod k).*

Theorem 4.2 immediately classifies reverse $6CS(v)$s:

Corollary 4.2 *A reverse $6CS(v)$ exists if and only if $v \equiv 1$ or 9 (mod 12).*

We will represent the 8-cycle



by any cyclic shift of (a,b,c,d,e,f,g,h) or (h,g,f,e,d,c,b,a). We now consider k-rotational $8CS(v)$s. Again, as with Lemma 4.1, a necessary condition for the existence of a k–rotational $8CS(v)$ is that k is even.

We now establish sufficiency for $k = 2$ in a series of lemmas.

Lemma 4.9 *If $v \equiv 1$ (mod 64) then there exists a 2-rotational $8CS(v)$.*

Proof. First suppose that $v = 65$. Consider the collection of 8-cycles:
$(\infty, 0_0, 4_0, 16_0, 31_1, 15_0, 0_1, 12_1)$, $(0_0, 8_0, 8_1, 0_1, 16_1, 24_1, 24_0, 16_0)$,
$(0_1, 4_1, 8_1, 12_1, 16_1, 20_1, 24_1, 28_1)$, $(0_0, 4_1, 8_0, 12_1, 16_0, 20_1, 24_0, 28_1)$,
$(0_0, 12_1, 24_0, 4_1, 16_0, 28_1, 8_0, 20_1)$, $(0_i, 3_i, 8_i, 14_i, 21_i, 2_i, 13_i, 23_i)$ for $i = 0,1$,
$(0_0, 1_0, 16_0, 17_1, 15_0, 14_1, 15_1, 30_1)$, $(0_0, 2_0, 16_0, 19_1, 14_0, 11_1, 13_1, 27_1)$,
$(0_0, 26_1, 3_0, 28_1, 4_0, 10_1, 2_0, 9_1)$, and $(0_0, 22_1, 4_0, 25_1, 6_0, 16_1, 3_0, 14_1)$.
Now suppose $v = 64t + 1$ where $t > 1$. Consider the collection of 8-cycles:
$(\infty, 0_0, (4t)_0, (12t)_0, (28t-1)_1, (12t-1)_0, (28t)_1, (20t)_1)$,
$(0_0, (12t)_0, (12t)_1, 0_1, (16t)_1, (28t)_1, (28t)_0, (16t)_0)$,
$(0_1, (4t)_1, (8t)_1, (12t)_1, (16t)_1, (20t)_1, (24t)_1, (28t)_1)$,
$(0_0, (4t)_1, (8t)_0, (12t)_1, (16t)_0, (20t)_1, (24t)_0, (28t)_1)$,
$(0_0, (12t)_1, (24t)_0, (4t)_1, (16t)_0, (28t)_1, (8t)_0, (20t)_1)$,
$(0_0, 1_0, (16t)_0, (16t+1)_1, (16t-1)_0, (16t-2)_1, (16t-1)_1, (32t-2)_1)$,
$(0_0, 2_0, (16t)_0, (16t+3)_1, (16t-1)_0, (16t-4)_1, (16t-2)_1, (32t-4)_1)$,
$(0_i, (3+4s)_i, (7+8s)_i, (12+12s)_i, (18+16s)_i, (16t+15+12s)_i, (11+8s)_i, (16t+6+4s)_i)$ for $s = 0,1,\ldots,t-2$ and $i = 0,1$,
$(0_i, (4t-1)_i, (8t)_i, (12t+2)_i, (16t+5)_i, (28t+6)_i, (8t+5)_i, (20t+3)_i)$ for $i = 0,1$,

$(0_i, (4t+4+4s)_i, (8t+9+8s)_i, (12t+15+12s)_i, (16t+22+16s)_i, (28t+18+12s)_i, (8t+13+8s)_i, (20t+7+4s)_i)$ for $s = 0, 1, \ldots, t-2$ and $i = 0, 1$,
$(0_0, (32t-5-4s)_1, 3_0, (32t-3-4s)_1, 4_0, (9+4s)_1, 2_0, (8+4s)_1)$ for $s = 0, 1, \ldots, t-3$ (omit if $t = 2$),
$(0_0, (28t+3)_1, 4_0, (28t+6)_1, 5_0, (4t+2)_1, 3_0, (4t+1)_1)$,
$(0_0, (28t-2-4s)_1, 3_0, (28t-4s)_1, 4_0, (4t+6+4s)_1, 2_0, (4t+5+4s)_1)$ for $s = 0, 1, \ldots, 2t-2$,
$(0_0, (20t+2)_1, 4_0, (20t+5)_1, 6_0, (12t+4)_1, 3_0, (12t+2)_1)$, and
$(0_0, (20t-3-4s)_1, 3_0, (20t-1-4s)_1, 4_0, (12t+7+4s)_1, 2_0, (12t+6+4s)_1)$ for $s = 0, 1, \ldots, t-2$.

In both cases, the blocks, along with their images under π_2, form a 2-rotational $8CS(v)$. ∎

Lemma 4.10 *If $v \equiv 17 \pmod{64}$ then there exists a 2-rotational $8CS(v)$.*

Proof. First suppose that $v = 17$. Consider the collection of 8-cycles:
$(\infty, 0_0, 2_0, 5_0, 1_1, 3_0, 3_1, 6_1)$, $(0_0, 1_0, 3_1, 1_1, 5_1, 7_1, 5_0, 4_0)$,
$(0_1, 1_1, 2_1, 3_1, 4_1, 5_1, 6_1, 7_1)$, $(0_0, 1_1, 2_0, 3_1, 4_0, 5_1, 6_0, 7_1)$, and
$(0_0, 3_1, 6_0, 1_1, 4_0, 7_1, 2_0, 5_1)$.

Now suppose $v = 64t + 17$ where $t > 0$. Consider the collection of 8-cycles:
$(\infty, 0_0, (4t+1)_0, (12t+3)_0, (28t+8)_1, (12t+4)_0, (28t+7)_1, (20t+5)_1)$,
$(0_0, (12t+3)_0, (12t+3)_1, 0_1, (16t+4)_1, (28t+7)_1, (28t+7)_0, (16t+4)_0)$,
$(0_1, (4t+1)_1, (8t+2)_1, (12t+3)_1, (16t+4)_1, (20t+5)_1, (24t+6)_1, (28t+7)_1)$,
$(0_0, (4t+1)_1, (8t+2)_0, (12t+3)_1, (16t+4)_0, (20t+5)_1, (24t+6)_0, (28t+7)_1)$,
$(0_0, (12t+3)_1, (24t+6)_0, (4t+1)_1, (16t+4)_0, (28t+7)_1, (8t+2)_0, (20t+5)_1)$,
$(0_i, (1+4s)_i, (3+8s)_i, (6+12s)_i, (10+16s)_i, (16t+13+12s)_i, (7+8s)_i, (16t+8+4s)_i)$ for $s = 0, 1, \ldots, t-1$ and $i = 0, 1$,
$(0_i, (4t+2+4s)_i, (8t+5+8s)_i, (12t+9+12s)_i, (16t+14+16s)_i, (28t+16+12s)_i, (8t+9+8s)_i, (20t+9+4s)_i)$ for $s = 0, 1, \ldots, t-1$ and $i = 0, 1$,
$(0_0, (32t+7-4s)_1, 3_0, (1-4s)_1, 4_0, (5+4s)_1, 2_0, (4+4s)_1)$ for $s = 0, 1, \ldots, t-1$,
$(0_0, (28t+6-4s)_1, 3_0, (28t+8-4s)_1, 4_0, (4t+6+4s)_1, 2_0, (4t+5+4s)_1)$ for $s = 0, 1, \ldots, 2t-1$,
$(0_0, (20t+6)_1, 4_0, (20t+8)_1, 5_0, (12t+7)_1, 2_0, (12t+6)_1)$, and
$(0_0, (20t+1-4s)_1, 3_0, (20t+3-4s)_1, 4_0, (12t+11+4s)_1, 2_0, (12t+10+4s)_1)$ for $s = 0, 1, \ldots, t-2$ (omit if $t = 1$).

In both cases, the blocks, along with their images under π_2, form a 2-rotational $8CS(v)$. ∎

Lemma 4.11 *If $v \equiv 33 \pmod{64}$ then there exists a 2-rotational $8CS(v)$.*

Proof. First suppose that $v = 33$. Consider the collection of 8-cycles:
$(\infty, 0_0, 2_0, 6_0, 13_1, 5_0, 14_1, 2_1)$, $(0_0, 6_0, 6_1, 0_1, 8_1, 14_1, 14_0, 8_0)$,
$(0_1, 2_1, 4_1, 6_1, 8_1, 10_1, 12_1, 14_1)$, $(0_0, 2_1, 4_0, 6_1, 8_0, 10_1, 12_0, 14_1)$,
$(0_0, 6_1, 12_0, 2_1, 8_0, 14_1, 4_0, 10_1)$, $(0_0, 1_0, 8_0, 9_1, 6_0, 5_1, 6_1, 13_1)$, and
$(0_0, 3_0, 8_0, 12_1, 7_0, 3_1, 6_1, 11_1)$.

Now suppose $v = 64t + 33$ where $t > 0$. Consider the collection of 8-cycles:
$(\infty, 0_0, (4t+2)_0, (12t+6)_0, (28t+13)_1, (12t+5)_0, (28t+14)_1, (20t+10)_1)$,

$(0_0,(12t+6)_0,(12t+6)_1,0_1,(16t+8)_1,(28t+14)_1,(28t+14)_0,(16t+8)_0)$,
$(0_1,(4t+2)_1,(8t+4)_1,(12t+6)_1,(16t+8)_1,(20t+10)_1,(24t+12)_1,(28t+14)_1)$,
$(0_0,(4t+2)_1,(8t+4)_0,(12t+6)_1,(16t+8)_0,(20t+10)_1,(24t+12)_0,(28t+14)_1)$,
$(0_0,(12t+6)_1,(24t+12)_0,(4t+2)_1,(16t+8)_0,(28t+14)_1,(8t+4)_0,(20t+10)_1)$,
$(0_0,1_0,(16t+8)_0,(16t+9)_1,(16t+7)_0,(16t+6)_1,(16t+7)_1,(32t+14)_1)$,
$(0_0,2_0,(16t+8)_0,(16t+11)_1,(16t+7)_0,(16t+4)_1,(16t+6)_1,(32t+12)_1)$,
$(0_i,(3+4s)_i,(7+8s)_i,(12+12s)_i,(18+16s)_i,(16t+23+12s)_i,(11+8s)_i,(16t+14+4s)_i)$ for $s=0,1,\ldots,t-2$ and $i=0,1$ (omit if $t=1$),
$(0_i,(4t-1)_i,(8t-1)_i,(12t)_i,(16t+3)_i,(28t+12)_i,(8t+4)_i,(20t+11)_i)$ for $i=0,1$,
$(0_i,(4t+4+4s)_i,(8t+9+8s)_i,(12t+15+12s)_i,(16t+22+16s)_i,(28t+26+12s)_i,(8t+13+8s)_i,(20t+15+4s)_i)$ for $s=0,1,\ldots,t-1$ and $i=0,1$,
$(0_0,(32t+11-4s)_1,3_0,(32t+13-4s)_1,4_0,(9+4s)_1,2_0,(8+4s)_1)$ for $s=0,1,\ldots,t-2$ (omit if $t=1$),
$(0_0,(28t+15)_1,4_0,(28t+17)_1,5_0,(4t+6)_1,2_0,(4t+5)_1)$,
$(0_0,(28t+10-4s)_1,3_0,(28t+12-4s)_1,4_0,(4t+10+4s)_1,2_0,(4t+9+4s)_1)$ for $s=0,1,\ldots,2t-1$, and
$(0_0,(20t+9-4s)_1,3_0,(20t+11-4s)_1,4_0,(12t+11+4s)_1,2_0,(12t+10+4s)_1)$ for $s=0,1,\ldots,t-1$.

In both cases, the blocks, along with their images under π_2, form a 2-rotational $8CS(v)$. ∎

Lemma 4.12 *If $v\equiv 49 \pmod{64}$ then there exists a 2-rotational $8CS(v)$.*

Proof. Suppose $v=64t+49$ where $t\geq 0$. Consider the collection of 8-cycles:
$(\infty,0_0,(4t+3)_0,(12t+9)_0,(28t+20)_1,(12t+8)_0,(28t+21)_1,(20t+15)_1)$,
$(0_0,(12t+9)_0,(12t+9)_1,0_1,(16t+12)_1,(28t+3)_1,(28t+3)_0,(16t+12)_0)$,
$(0_1,(4t+3)_1,(8t+6)_1,(12t+9)_1,(16t+12)_1,(20t+15)_1,(24t+18)_1,(28t+21)_1)$,
$(0_0,(4t+3)_1,(8t+6)_0,(12t+9)_1,(16t+12)_0,(20t+15)_1,(24t+18)_0,(28t+21)_1)$,
$(0_0,(12t+9)_1,(24t+18)_0,(4t+3)_1,(16t+12)_0,(28t+21)_1,(8t+6)_0,(20t+15)_1)$,
$(0_i,(1+4s)_i,(3+8s)_i,(6+12s)_i,(10+16s)_i,(16t+21+12s)_i,(7+8s)_i,(16t+16+4s)_i)$ for $s=0,1,\ldots,t-1$ and $i=0,1$ (omit if $t=0$),
$(0_i,(4t+1)_i,(8t+3)_i,(12t+7)_i,(16t+12)_i,(28t+23)_i,(8t+9)_i,(20t+17)_i)$ for $i=0,1$,
$(0_i,(4t+6+4s)_i,(8t+13+8s)_i,(12t+21+12s)_i,(16t+30+16s)_i,(28t+36+12s)_i,(8t+17+8s)_i,(20t+21+4s)_i)$ for $s=0,1,\ldots,t-1$ and $i=0,1$ (omit if $t=0$),
$(0_0,(32t+23-4s)_1,3_0,(32t+25-4s)_1,4_0,(5+4s)_1,2_0,(4+4s)_1)$ for $s=0,1,\ldots,t-1$ (omit if $t=0$),
$(0_0,(28t+23)_1,4_0,(28t+26)_1,6_0,(4t+7)_1,3_0,(4t+5)_1)$,
$(0_0,(28t+18-4s)_1,3_0,(28t+20-4s)_1,4_0,(4t+10+4s)_1,2_0,(4t+9+4s)_1)$ for $s=0,1,\ldots,2t-1$ (omit if $t=0$),
$(0_0,(20t+18)_1,4_0,(20t+21)_1,5_0,(12t+11)_1,3_0,(12t+10)_1)$, and
$(0_0,(20t+13-4s)_1,3_0,(20t+15-4s)_1,4_0,(12t+15+4s)_1,2_0,(12t+14+4s)_1)$ for $s=0,1,\ldots,t-1$ (omit if $t=0$).

These blocks, along with their images under π_2, form a 2-rotational $8CS(v)$. ∎

As in Theorem 4.1, the results of Lemmas 4.9 to 4.12 allow us to establish necessary and sufficient conditions for the existence of a k-rotational $8CS(v)$, for all k.

Theorem 4.3 *A k-rotational $8CS(v)$ exists if and only if $v \equiv 1 \pmod{16}$, k is even and $v \equiv 1 \pmod{k}$.*

Theorem 4.3 immediately classifies reverse $8CS(v)$s:

Corollary 4.3 *A reverse $8CS(v)$ exists if and only if $v \equiv 1 \pmod{16}$.*

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