Decompositions of the Complete Digraph and the Complete Graph which admit Certain Automorphisms

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Abstract. Decompositions of the complete digraph into isomorphic copies of orientations of 4-cycles and 5-cycles are considered. Necessary and sufficient conditions for such decompositions admitting cyclic or rotational automorphisms are given. Some new decompositions of the complete digraph into certain (non-self-converse) orientations of the 6-cycle are also given. The spectrum of 4-, 6-, and 8-cycle systems admitting either a reverse automorphism or a k-rotational automorphism for any k are determined.

1 Introduction

Let $D_v$ denote the complete digraph on $v$ vertices. If $g$ is a digraph, then a $g$-decomposition of $D_v$ is a set $\gamma = \{g_1, g_2, \ldots, g_n\}$ of arc-disjoint subgraphs of $D_v$ each of which is isomorphic to $g$ and such that $\bigcup_{i=1}^{n} A(g_i) = A(D_v)$, where $A(G)$ is the arc set of digraph $G$. Several of these decompositions are equivalent to block designs. For example, a $D_3$-decomposition of $D_v$ is equivalent to a Steiner triple system of order $v$. A $k$-circuit decomposition of $D_v$ is equivalent to a $k$-Mendelsohn design of order $v$, denoted $M(k, v)$.

There are two orientations of the 3-cycle: the 3-circuit and the digraph (called a "transitive triple"):

\begin{center}
\begin{tikzpicture}
\node (a) at (0,0) [circle,fill,inner sep=2pt] {}; \node (b) at (1,0) [circle,fill,inner sep=2pt] {}; \node (c) at (0.5,0.866) [circle,fill,inner sep=2pt] {};
\draw (a) to (b); \draw (b) to (c); \draw (c) to (a);
\end{tikzpicture}
\end{center}

A decomposition of $D_v$ into 3-circuits is equivalent to a Mendelsohn triple system of order $v$ (or a $M(3, v)$) which exists if and only if $v \equiv 0$ or 1 (mod 3), $v \neq 6$ [14]. A

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decomposition of $D_v$ into transitive triples is equivalent to a directed triple system of order $v$, which exists if and only if $v \equiv 0$ or $1 \pmod{3}$ [11].

There are four orientations of the 4-cycle: the 4-circuit and the following

\[ \begin{array}{ccc}
    b & c \\
    a & d \\
\end{array} \quad \begin{array}{ccc}
    b & c \\
    a & d \\
\end{array} \quad \begin{array}{ccc}
    b & c \\
    a & d \\
\end{array} \]

We represent $X$ as $[a, b, c, d]_x$, $Y$ as $[a, b, c, d]_y$, and $Z$ as $[a, b, c, d]_z$. A $M(4, v)$ exists if and only if $v \equiv 0$ or $1 \pmod{4}$, $v \neq 4 [23]$. A $X$-decomposition of $D_v$ exists if and only if $v \equiv 0$ or $1 \pmod{4}$, $v \neq 5$, a $Y$-decomposition of $D_v$ exists if and only if $v \equiv 0$ or $1 \pmod{4}$, $v \notin \{4, 5\}$, and a $Z$-decomposition of $D_v$ exists if and only if $v \equiv 1 \pmod{4}$ [10].

There are four orientations of the 5-cycle: the 5-circuit and the following

\[ \begin{array}{ccc}
    b & c & d \\
    a & e \\
\end{array} \quad \begin{array}{ccc}
    b & c & d \\
    a & e \\
\end{array} \quad \begin{array}{ccc}
    b & c & d \\
    a & e \\
\end{array} \]

We represent $A_i$ as $[a, b, c, d, e]_i$, for $i = 1, 2, 3$. A $M(5, v)$ exists if and only if $v \equiv 0$ or $1 \pmod{5}$ [2]. This is also the spectrum of an $A_i$-decomposition of $D_v$ for $i = 1, 2, 3 [1]$. A digraph $d$ is said to be self-converse if reversing each of the arcs produces a digraph isomorphic to $d$. Each of the orientations of the 3-cycle, 4-cycle and 5-cycle are self-converse. These are the only cycles for which each orientation is self-converse [9]. Varma [26] gave necessary and sufficient conditions for the decomposition of $D_v$ into self-converse orientations of 6-cycles, 7-cycles and 8-cycles.

An automorphism of a digraph decomposition of $D_v$ is a permutation of the vertex set of $D_v$ which fixes the collection of isomorphic digraphs in the decomposition. A decomposition of $D_v$ admitting an automorphism consisting of a cycle of length $v$ is said to be cyclic. A decomposition of $D_v$ admitting an automorphism consisting of a fixed point and a cycle of length $v - 1$ is said to be rotational. Necessary and sufficient conditions are known for the existence of cyclic $M(k, v)$s for $k = 3, 4, 5, 6, 7, 8 [6, 15, 16]$. Necessary and sufficient conditions are known for the existence of rotational $M(k, v)$s for $k = 3, 4, 5 [4, 18]$. Cyclic directed triple systems exist if and only if $v \equiv 1, 4, 7 \pmod{12}$ [7] and rotational directed triple systems exist if and only if $v \equiv 0 \pmod{3}$ [5]. The purpose of this paper is to explore cyclic and rotational automorphisms of decompositions of $D_v$ into orientations of 4-cycles and 5-cycles. In addition, we
address the existence question for decompositions of $D_v$ into copies of the non-self-converse orientations of the 6-cycle. Finally, we present some related results for cycle systems (i.e. decompositions of the complete graph into cycles of a given length).

2 Cyclic Decompositions

In this section, we give necessary and sufficient conditions for the existence of cyclic decompositions of $D_v$ into orientations of 4-cycles and 5-cycles. Throughout this section, we assume $D_v$ has vertex set $\mathbb{Z}_v$ and the automorphism is the permutation $\alpha = (0, 1, \ldots, v - 1)$. With each arc $(a, b)$ of $D_v$ we associate a difference of $b - a \pmod{v}$. The existence of a cyclic $X$-decomposition and of a cyclic $Y$-decomposition of $D_v$ implies a partitioning of the set of differences $\{1, 2, \ldots, v - 1\}$ into difference 4-tuples $(d_i, d_j, d_k, d_l)$ such that

\[ d_i + d_j + d_k \equiv d_l \pmod{v} \text{ for } X\text{-decompositions, or} \]
\[ d_i + d_j \equiv d_k + d_l \pmod{v} \text{ for } Y\text{-decompositions.} \]

Therefore, a necessary condition for either such decomposition is that $v - 1 \equiv 0 \pmod{4}$. We show this condition is sufficient in the next two theorems.

Theorem 2.1 A cyclic $X$-decomposition of $D_v$ exists if and only if $v \equiv 1 \pmod{4}$, $v \neq 5$.

Proof. We consider two cases.

Case 1. Suppose $v \equiv 1 \pmod{8}$, say $v = 8t + 1$. Consider the blocks:

- $[0, 1 + i, 6t + 2 + 2i, 2t + 1 + i]_x$ for $i = 0, 1, \ldots, t - 1$, and
- $[0, t + 1 + i, 1 + 2t, 5t + 1 + i]_x$ for $i = 0, 1, \ldots, t - 1$.

Case 2. Suppose $v \equiv 5 \pmod{8}$, say $v = 8t + 5$, $t > 0$. Consider the blocks:

- $[0, 1, 2t + 3, 4t + 6]_x$; $[0, 2 + i, 1, 2t + 5 + 2i]_x$ for $i = 0, 1, \ldots, t$, and
- $[0, t + 3 + i, 1, 4t + 8 + 2i]_x$ for $i = 0, 1, \ldots, t - 2$ (omit if $t = 1$).

In both cases, the blocks, along with their images under $\alpha$, form a cyclic $X$-decomposition of $D_v$.

Theorem 2.2 A cyclic $Y$-decomposition of $D_v$ exists if and only if $v \equiv 1 \pmod{4}$, $v \neq 5$.

Proof. Suppose $v = 4t + 1$, $t > 1$. Consider the blocks:

- $[0, 2t - 1, 2t - 2, 4t - 1]_y$, and $[0, 1 + i, 4t - 1, 2t + 1 + i]_y$ for $i = 0, 1, \ldots, t - 2$.

These blocks, along with their images under $\alpha$, form a cyclic $Y$-decomposition of $D_v$.

Theorem 2.3 A cyclic $Z$-decomposition of $D_v$ exists if and only if $v \equiv 1 \pmod{4}$.

Proof. The necessary conditions follow from the spectrum of a $Z$-decomposition of $D_v$. Suppose $v \equiv 1 \pmod{4}$, say $v = 4t + 1$. Consider the blocks:

- $[0, 1 + i, 2t + 2 + 2i, 2t + 1 + i]_x$ for $i = 0, 1, \ldots, t - 1$.

These blocks, along with their images under $\alpha$, form a cyclic $Z$-decomposition of $D_v$. 
We now turn our attention to the existence of cyclic $A_i$-decompositions of $D_v$ for $i = 1, 2, 3$. We have as a necessary condition:

**Lemma 2.1** If a cyclic $A_i$-decomposition of $D_v$ exists, where $i = 1, 2, 3$, then $v \equiv 1, 11, \text{ or } 16 \pmod{20}$.

**Proof.** The existence of such a decomposition implies a partitioning of the set of differences $\{1, 2, \ldots, v - 1\}$ into difference 5-tuples $(d_i, d_j, d_k, d_l, d_m)$ such that

$$d_i + d_j + d_k \equiv d_l + d_m \pmod{v}$$

for $A_1$- and $A_2$-decompositions, or

$$d_i + d_j + d_k + d_l \equiv d_m \pmod{v}$$

for $A_3$-decompositions.

Therefore, $v - 1 \equiv 0 \pmod{5}$ is necessary. Notice that from the above condition on the difference 5-tuples, we can partition the difference set into two sets $A$ and $B$ such that the sum of the elements of $A$ is congruent to the sum of the elements of $B$ modulo $v$. That is, there exists $a$ and $b$ such that $a + b = \frac{v(v - 1)}{2}$ and $a \equiv b \pmod{v}$. But if $v \equiv 6 \pmod{20}$, say $v = 20t + 6$, then $a + b = (20t + 5)(10t + 3)$ which is odd. But if $a \equiv b \pmod{20t + 6}$ then $a + b$ is even. Therefore $v \equiv 6 \pmod{20}$ is not possible and the necessary conditions for such a system are $v \equiv 1, 11, \text{ or } 16 \pmod{20}$.

We now show the necessary conditions of Lemma 2.1 are also sufficient.

**Theorem 2.4** A cyclic $A_1$-decomposition of $D_v$ exists if and only if $v \equiv 1, 11, \text{ or } 16 \pmod{5}$.

**Proof.** We consider two cases.

**Case 1.** Suppose $v \equiv 1 \pmod{10}$, say $v = 10t + 1$. Consider the blocks:

$$[0, 1 + i, 5t + 1, 4t - i, 4t - 1 - 2i]_1$$

for $i = 0, 1, \ldots, t - 1$, and

$$[0, 2t + 2 + 2i, 7t + 3 + 3i, 5t + 4 + 4i, 6t + 2 + 2i]_1$$

for $i = 0, 1, \ldots, t - 1$.

**Case 2.** Suppose $v \equiv 16 \pmod{20}$, say $v = 20t + 16$. Consider the blocks:

$$[0, 20t + 14 - 2i, 7t + 4 - i, 20t + 15, 1 + 2i]_1$$

for $i = 0, 1, \ldots, 2t - 1$ (omit if $t = 0$),

$$[0, 4t + 3, 13t + 9, 8t + 4, 4t + 1]_1$$

for $i = 0, 1, \ldots, 9t + 6, 5t + 4]_1$,

$$[0, 10t + 8 + 9, 3t + 3, 10t + 8 - i, 20t + 15 - 2i]_1$$

for $i = 0, 1, \ldots, t$, and

$$[0, 5t + 3 - i, 17t + 12 - 2i, 19t + 15, 14t + 12 + 2i]_1$$

for $i = 0, 1, \ldots, t - 1$ (omit if $t = 0$).

In both cases, the blocks, along with their images under $\alpha$, form a cyclic $A_1$-decomposition of $D_v$.

**Theorem 2.5** A cyclic $A_2$-decomposition of $D_v$ exists if and only if $v \equiv 1, 11, \text{ or } 16 \pmod{5}$.

**Proof.** We consider two cases.

**Case 1.** Suppose $v \equiv 1 \pmod{10}$, say $v = 10t + 1$. Consider the blocks:

$$[0, 1 + i, 5t + 1, 5t - i, 4t - 1 - 2i]_2$$

for $i = 0, 1, \ldots, t - 1$, and

$$[0, 2t + 2 + 2i, 7t + 3 + 3i, 5t + 1 + i, 6t + 2 + 2i]_2$$

for $i = 0, 1, \ldots, t - 1$.

**Case 2.** Suppose $v \equiv 16 \pmod{20}$, say $v = 20t + 16$. Consider the blocks:

$$[0, 20t + 14 - 2i, 7t + 4 - i, 20t + 15, 1 + 2i]_2$$

for $i = 0, 1, \ldots, 2t - 1$ (omit if $t = 0$),

$$[0, 4t + 3, 13t + 9, 8t + 4, 4t + 1]_2$$

for $i = 0, 1, \ldots, 9t + 6, 5t + 4]_2$,

$$[0, 16t + 14, 11t + 10, 15t + 12, 5t + 4]_2$$,

$$[0, 4t + 3, 13t + 9, 8t + 4, 4t + 1]_2$$

for $i = 0, 1, \ldots, 9t + 6, 5t + 4]_2$,

$$[0, 16t + 14, 11t + 10, 15t + 12, 5t + 4]_2$$.

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\[ [0, 10t + 9 + i, 3t + 3, 13t + 10 - i, 13t + 11 + i]_2 \text{ for } i = 0, 1, \ldots, t, \text{ and} \]
\[ [0, 15t + 13 + i, 7t + 6, 12t + 9 - i, 18t + 13 - 2i]_3 \text{ for } i = 0, 1, \ldots, t - 1 \text{ (omit if } t = 0). \]

In both cases, the blocks, along with their images under \( \alpha \), form a cyclic \( A_2 \)-decomposition of \( D_v \).

**Theorem 2.6** A cyclic \( A_3 \)-decomposition of \( D_v \) exists if and only if \( v \equiv 1, 11, \text{ or } 16 \pmod{5} \).

**Proof.** We consider two cases.

**Case 1.** Suppose \( v \equiv 1 \pmod{10} \), say \( v = 10t + 1 \). Consider the blocks:
\[ [0, 1 + i, t + 2 + 2i, 5t + 1, 5t - i]_3 \text{ for } i = 0, 1, \ldots, t - 1, \text{ and} \]
\[ [0, 2t + 2 + 2i, t + 1 + i, 7t + 3 + 3i, 5t + 1 + i]_3 \text{ for } i = 0, 1, \ldots, t - 1. \]

**Case 2.** Suppose \( v \equiv 16 \pmod{20} \), say \( v = 20t + 16 \). Consider the blocks:
\[ [0, 1 + 2i, 20t + 15, 7t + 4 - i, 7t + 6 + i]_3 \text{ for } i = 0, 1, \ldots, 2t - 1 \text{ (omit if } t = 0), \]
\[ [0, 4t + 1, 8t + 4, 13t + 9, 9t + 6]_3, [0, 5t + 4, 9t + 6, 19t + 14, 15t + 12]_3, \]
\[ [0, 20t + 15 - 2i, 10t + 8 - i, 3t + 3, 13t + 10 - i]_3 \text{ for } i = 0, 1, \ldots, t, \text{ and} \]
\[ [0, 14t + 12 + i, 19t + 15, 17t + 12 - 2i, 12t + 9 - i]_3 \text{ for } i = 0, 1, \ldots, t - 1 \text{ (omit if } t = 0). \]

In both cases, the blocks, along with their images under \( \alpha \), form a cyclic \( A_3 \)-decomposition of \( D_v \).

We now consider the cyclic decomposition of \( D_v \) into the following orientations of the 6-cycle:

We represent \( B_1 \) as \([a, b, c, d, e, f]_1 \) and \( B_2 \) as \([a, b, c, d, e, f]_2 \). Neither of these orientations is self-converse. However, \( B_1 \) is the converse of \( B_2 \). Necessary and sufficient conditions are known for the decompositions of \( D_v \) into the remaining orientations of the 6-cycle (each of which is self-converse) [26]. Necessary conditions for the existence of \( B_1 \)- or \( B_2 \)-decompositions of \( D_v \) are that \( v \equiv 0, 1, \text{ or } 3 \pmod{6} \).

**Theorem 2.7** A cyclic \( B_1 \)-decomposition of \( D_v \) exists if and only if \( v \equiv 1 \pmod{6} \).

**Proof.** As above, a necessary condition for such a system is that the number of differences be divisible by 6. That is, \( v \equiv 1 \pmod{6} \) is necessary. We show sufficiency in three cases.

**Case 1.** Suppose \( v \equiv 1 \pmod{12} \), say \( v = 12t + 1 \). Consider the blocks:
\[ [0, 1 + 12i, 4 + 24i, 2 + 12i, 9 + 24i, 5 + 12i]_1 \text{ for } i = 0, 1, \ldots, t - 1, \text{ and} \]
\[ [0, 6 + 12i, 16 + 24i, 8 + 12i, 20 + 24i, 11 + 12i]_1 \text{ for } i = 0, 1, \ldots, t - 1. \]

**Case 2.** Suppose \( v \equiv 7 \pmod{24} \), say \( v = 24t + 7 \). Consider the blocks:
\[ [0, 1 + 12i, 6 + 24i, 3 + 12i, 10 + 24i, 4 + 12i]_1 \text{ for } i = 0, 1, \ldots, t - 1 \text{ (omit if } t = 0), \]

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\[ [0, 8 + 12i, 18 + 24i, 9 + 12i, 23 + 24i, 12 + 12i] \text{ for } i = 0, 1, \ldots, t - 1 \text{ (omit if } t = 0), \]
\[ [0, 12t + 5 + 12i, 24t + 14 + 24i, 12t + 7 + 12i, 24t + 18 + 24i, 12t + 8 + 12i] \text{ for } i = 0, 1, \ldots, t - 1 \text{ (omit if } t = 0), \]
\[ [0, 12t + 12 + 12i, 24t + 26 + 24i, 12t + 13 + 12i, 24t + 31 + 24i, 12t + 16 + 12i] \text{ for } i = 0, 1, \ldots, t - 1 \text{ (omit if } t = 0), \text{ and } [0, 2, 12t + 6, 3, 12t + 4, 24t + 5]. \text{]

Case 3. Suppose \( v \equiv 19 \pmod{24} \), say \( v = 24t + 19 \). Consider the blocks:
\[ [0, 1 + 12i, 6 + 24i, 3 + 12i, 10 + 24i, 4 + 12i] \text{ for } i = 0, 1, \ldots, t, \]
\[ [0, 8 + 12i, 18 + 24i, 9 + 12i, 23 + 24i, 12 + 12i] \text{ for } i = 0, 1, \ldots, t - 1 \text{ (omit if } t = 0), \]
\[ [0, 12t + 12 + 12i, 24t + 26 + 24i, 12t + 13 + 12i, 24t + 31 + 24i, 12t + 16 + 12i] \text{ for } i = 0, 1, \ldots, t - 1 \text{ (omit if } t = 0), \text{ and } [0, 2, 12t + 11, 1, 12t + 9, 24t + 17]. \text{]

In each case, the blocks, along with their images under \( \alpha \), form a cyclic \( B_1 \)-decomposition of \( D_v \).

Since \( B_2 \) is the converse of \( B_1 \), the existence of a cyclic \( B_1 \)-decomposition of \( D_v \) implies the existence of a cyclic \( B_2 \) decomposition of \( D_v \) (and conversely). We therefore have:

**Theorem 2.8** A cyclic \( B_2 \)-decomposition of \( D_v \) exists if and only if \( v \equiv 1 \pmod{6} \).

### 3 Rotational Decompositions

In this section, we give necessary and sufficient conditions for the existence of rotational decompositions of \( D_v \) into orientations of 4-cycles and 5-cycles. Throughout this section, we assume \( D_v \) has vertex set \( \{\infty\} \cup \mathbb{Z}_{v-1} \) and the automorphism is the permutation \( \beta = (\infty)(0, 1, \ldots, v - 2) \).

**Lemma 3.1** If a rotational \( X \)- or \( Y \)-decomposition of \( D_v \) exists, then \( v \equiv 0 \pmod{4} \).

**Proof.** The existence of a rotational \( X \)- or \( Y \)-decomposition of \( D_v \) implies the partitioning of the set of differences \( \{1, 2, \ldots, v - 2\} \setminus \{d_1, d_2\} \), where \( d_1 \) and \( d_2 \) are two differences, into difference 4-tuples \( (d_i, d_j, d_k, d_l) \) such that
\( d_i + d_j + d_k \equiv d_l \pmod{v} \) for \( X \)-decompositions, or
\( d_i + d_j \equiv d_k + d_l \pmod{v} \) for \( Y \)-decompositions.

Therefore, a necessary condition for either such decomposition is that \( v - 4 \equiv 0 \pmod{4} \).

We show this condition is sufficient in the next two theorems.

**Theorem 3.1** A rotational \( X \)-decomposition of \( D_v \) exists if and only if \( v \equiv 0 \pmod{4} \).

**Proof.** We consider two cases.

**Case 1.** Suppose \( v \equiv 0 \pmod{8} \), say \( v = 8t \). Consider the blocks:
\[ [6t - 1, \infty, 0, 6t - 2] \text{, } [0, 1 + i, 6t + 2i, 2t + i] \text{ for } i = 0, 1, \ldots, t - 1, \text{ and } \]
\[ [6t + 1, \infty, 0, 6t + 2] \text{, } [0, 1 - i, 6t + 2i, 2t + i] \text{ for } i = 0, 1, \ldots, t - 1, \text{ and } \]
\[ 0, t + 1 + i, 8t + 2i, 5t - 1 + i \] for \( i = 0, 1, \ldots, t - 2 \) (omit if \( t = 1 \)).

Case 2. Suppose \( v \equiv 4 \pmod{8} \), say \( v = 8t + 4 \). Consider the blocks:
\[ [1, \infty, 6t + 2, 0, 1 + i, 6t + 4 + 2i, 2t + 1 + i]_x \] for \( i = 0, 1, \ldots, t - 1 \) (omit if \( t = 0 \)), and
\[ [0, t + 1 + i, 8t + 4 + 2i, 5t + 1 + i]_x \] for \( i = 0, 1, \ldots, t - 1 \) (omit if \( t = 0 \)).

In both cases, the blocks, along with their images under the permutation \( \beta \), form a rotational \( X \)-decomposition of \( D_v \).

**Theorem 3.2** A rotational \( Y \)-decomposition of \( D_v \) exists if and only if \( v \equiv 0 \pmod{4} \), \( v \neq 4 \).

**Proof.** Suppose \( v \equiv 0 \pmod{4} \), say \( v = 4t, t \geq 2 \). Consider the blocks:
\[ [1, \infty, 4t - 3, 0]_y \] and \( [0, 1 + i, 4t - 3, 2t - 1 + i]_y \) for \( i = 0, 1, \ldots, t - 2 \).

These blocks along with their images under the permutation \( \pi = (\infty)(0, 1, \ldots, 4t-2) \), form a \( Y \)-decomposition of \( D_v \) where the point set of \( D_v \) is \( \{\infty\} \cup \mathbb{Z}_{4t-1} \).

We note that Theorems 2.1, 2.2, 3.1, and 3.2 combine to give direct constructions of \( X \), \( Y \), and \( Z \)-decompositions of \( D_v \) for all admissible \( v \).

Concerning rotational \( Z \)-decompositions of \( D_v \) we have:

**Theorem 3.3** A rotational \( Z \)-decomposition of \( D_v \) does not exist for any \( v \).

**Proof.** Such a system must have a block containing the fixed point \( \infty \). So there must be either a block of the form \( A = [a, \infty, b, c]_x \) or \( B = [c, a, \infty, b]_x \). If we apply \( \beta^{b-a} \) to block \( A \), we get the arc \( (a, \infty) \) twice in the decomposition, a contradiction. If we apply \( \beta^{b-a} \) to block \( B \), we get the arc \( (\infty, a) \) twice in the decomposition, another contradiction. Therefore, a rotational \( Z \)-decomposition of \( D_v \) does not exist.

We now turn our attention to rotational \( A_i \)-decompositions of \( D_v \) for \( i = 1, 2, 3 \).

**Lemma 3.2** If a rotational \( A_i \)-decomposition of \( D_v \) exists for \( i = 1, 2 \) or 3, then \( v \equiv 0 \pmod{5} \).

**Proof.** As in Lemma 3.1, the existence of such a decomposition implies a partitioning of the set \( \{1, 2, \ldots, v - 2\} \setminus \{d_1, d_2, d_3\} \) into difference 5-tuples such that
\[ d_i + d_j + d_k \equiv d_i + d_m \pmod{v - 1} \] for \( i = 1, 2 \), or
\[ d_i + d_j + d_k + d_l \equiv d_m \pmod{v - 1} \] for \( i = 3 \).

Therefore, a necessary condition for such a system is that \( v - 5 \equiv 0 \pmod{5} \).

We show this condition is sufficient in the next three theorems.

**Theorem 3.4** A rotational \( A_1 \)-decomposition of \( D_v \) exists if and only if \( v \equiv 0 \pmod{5} \).

**Proof.** We consider two cases.

Case 1. Suppose \( v \equiv 0 \pmod{10} \), say \( v = 10t \). Consider the blocks:
\[ [0, 1 + i, 4t + 1 + 2i, 7t + 1 + 3i, 7t + 2i]_1 \] for \( i = 0, 1, \ldots, t - 1 \),
\[ [0, t + 2 + 2i, 7t + i, 4t - 1, 3t - 3 - 2i]_1 \text{ for } i = 0, 1, \ldots, t - 2 \text{ (omit if } t = 1) \text{; and } [0, \infty, 6t + 1, 3t]. \]

Case 2. If \( v = 5 \), consider the block \([0, \infty, 2, 3, 1]_1\). Next, suppose \( v \equiv 5 \pmod{10} \), say \( v = 10t + 5 \), \( t \geq 1 \). Consider the blocks:

\[ [0, 6t + 3 - i, 9t + 4, 6t + 4 + i, 1 + 2i]_1 \text{ for } i = 0, 1, \ldots, t - 1, \]
\[ [0, 4 + 2i, 7t + 7 + i, 3, 10t + 3 - 2i]_1 \text{ for } i = 0, 1, \ldots, t - 2 \text{ (omit if } t = 1) \text{,} \]
\[ [0, 6t + 4, t + 1, 3t + 2, 8t + 5]_1, \text{ and } [0, \infty, 5t + 2, 5t, 10t + 2]. \]

These blocks, along with their images under \( \beta \), form a rotational \( A_1 \)-decomposition of \( D_v \).

**Theorem 3.5** A rotational \( A_2 \)-decomposition of \( D_v \) exists if and only if \( v \equiv 0 \pmod{5} \).

**Proof.** We consider two cases.

**Case 1.** Suppose \( v \equiv 0 \pmod{10} \), say \( v = 10t \). Consider the blocks:

\[ [0, 1 + i, 4t + 1 + 2i, 4t + i, 7t + 2i]_2 \text{ for } i = 0, 1, \ldots, t - 1, \]
\[ [0, t + 2 + 2i, 7t + i, 6t - 2 - i, 3t - 3 - 2i]_2 \text{ for } i = 0, 1, \ldots, t - 2 \text{ (omit if } t = 1) \text{,} \]
\[ [0, 3t - 1, \infty, 9t - 1, 3t]_2. \]

**Case 2.** Suppose \( v \equiv 5 \pmod{10} \), say \( v = 10t + 5 \). Consider the blocks:

\[ [0, 3t + 1 + i, 9t + 4, 3t + 1 + i, 3t - i]_2 \text{ for } i = 0, 1, \ldots, t - 1, \]
\[ [0, 7t + 3 - i, 7t + 7 + i, 7t + 3 - i, 10t + 3 - 2i]_2 \text{ for } i = 0, 1, \ldots, t - 2 \text{ (omit if } t = 1) \text{,} \]
\[ [0, 6t + 4, t + 1, 6t + 4, 8t + 5]_2, \text{ and } [0, \infty, 5t - 2, 5t, 10t + 2]_2. \]

These blocks, along with their images under \( \beta \), form a rotational \( A_2 \)-decomposition of \( D_v \).

**Theorem 3.6** A rotational \( A_3 \)-decomposition of \( D_v \) exists if and only if \( v \equiv 0 \pmod{5} \).

**Proof.** We consider two cases.

**Case 1.** Suppose \( v \equiv 0 \pmod{10} \), say \( v = 10t \). Consider the blocks:

\[ [0, 6t - 1, \infty, 1, 3t]_3, [0, 1 + i, 7t, 4t + 1 + 2i, 4t + i]_3 \text{ for } i = 0, 1, \ldots, t - 1, \]
\[ [0, t + 2 + 2i, 4t + 3 + 3i, 7t + i, 6t - 2 - i]_3 \text{ for } i = 0, 1, \ldots, t - 2 \text{ (omit if } t = 1) \text{.} \]

**Case 2.** If \( v = 5 \), consider the block \([0, 1, \infty, 3, 2]_3\). Next, suppose \( v \equiv 5 \pmod{10} \), say \( v = 10t + 5 \), \( t \geq 1 \). Consider the blocks:

\[ [0, 1 + 2i, 6t + 4 + i, 9t + 4, 3t + 1 + i]_3 \text{ for } i = 0, 1, \ldots, t - 1, \]
\[ [0, 10t - 3 - 2i, 10t + 7, 7t + 7 + i, 7t + 3 - i]_3 \text{ for } i = 0, 1, \ldots, t - 2 \text{ (omit if } t = 1) \text{,} \]
\[ [0, 8t + 3, 3t, 8t + 3, 6t + 4]_3, \text{ and } [0, \infty, 5t - 2, 5t, 10t + 2]_3. \]

These blocks, along with their images under \( \beta \), form a rotational \( A_3 \)-decomposition of \( D_v \).

We note that Theorems 2.4, 2.5, 2.6, 3.4, 3.5 and 3.6 combine to give direct constructions of \( A_v \)-decompositions of \( D_v \) for all admissible \( v \) except for \( v \equiv 6 \pmod{20} \).

We now consider \( B_2 \)- and \( B_2 \)-decompositions of \( D_v \) which admit a rotational automorphism. We have:
Theorem 3.7 A rotational $B_1$-decomposition of $D_v$ exists if and only if $v \equiv 0 \pmod{6}$.

Proof. An argument similar to those in Lemmas 3.1 and 3.2 shows that $v \equiv 0 \pmod{6}$ is necessary. For sufficiency, we consider two cases.

Case 1. Suppose $v \equiv 0 \pmod{12}$, say $v = 12t$. Consider the blocks:

$[0, 1 + 2i, 4 + 24i, 2 + 12i, 9 + 24i, 5 + 12i]$, for $i = 0, 1, \ldots, t - 1$,

$[0, 6 + 12i, 16 + 24i, 8 + 12i, 20 + 24i, 11 + 12i]$, for $i = 0, 1, \ldots, t - 2$ (omit if $t = 1$),

and $[0, \infty, 12t - 6, 12t - 3, 12t - 5, 12t - 2]$.

Case 2. If $v = 6$, consider the block $[0, \infty, 3, 1, 2, 4]$. Next suppose $v \equiv 6 \pmod{12}$, say $v = 12t + 6$, $t \geq 1$. Consider the blocks:

$[0, 1 + 12i, 4 + 24i, 2 + 12i, 9 + 24i, 5 + 12i]$, for $i = 0, 1, \ldots, t - 1$,

$[0, 6 + 12i, 16 + 24i, 8 + 12i, 20 + 24i, 11 + 12i]$, for $i = 0, 1, \ldots, t - 1$, and

$[0, \infty, 12t - 3, 12t - 6, 12t - 6, 12t - 1, 12t + 1]$.

In both cases, the blocks, along with their images under $\beta$, form a rotational $B_1$-decomposition of $D_v$.

Since $B_2$ is the converse of $B_1$, the existence of a rotational $B_1$-decomposition of $D_v$ implies the existence of a rotational $B_2$-decomposition of $D_v$ (and conversely). We therefore have:

Theorem 3.8 A rotational $B_2$-decomposition of $D_v$ exists if and only if $v \equiv 0 \pmod{6}$.

We note that Theorems 2.7, 2.8, 3.7, and 3.8 combine to give the existence of a $B_1$- or $B_2$-decomposition of $D_v$ when $v \equiv 0$ or $1 \pmod{6}$. We leave the case $v \equiv 3 \pmod{6}$ open.

4 Related Results For Cycle Systems

The decomposition of a graph is defined similarly to the decomposition of a digraph, and an automorphism of a graph decomposition is analogous to an automorphism of a digraph decomposition. An $n$-cycle system of order $v$, denoted $nCS(v)$, is a decomposition of $K_v$ (the complete graph on $v$ vertices) into cycles of length $n$. A $4CS(v)$ exists if and only if $v \equiv 1 \pmod{8}$ [12], a $6CS(v)$ exists if and only if $v \equiv 1$ or $19 \pmod{12}$ [20], and an $8CS(v)$ exists if and only if $v \equiv 1 \pmod{16}$ [12]. For a survey of results on cycle systems see [13]. A cyclic 3-cycle system (or Steiner triple system) exists if and only if $v \equiv 1$ or $3 \pmod{6}$, $v \neq 9$ [17]. Cyclic cycle systems in general are explored in [20] and [21]. We slightly generalize the idea of a rotational automorphism by defining a $k$-rotational automorphism as one consisting of a fixed point and $k$ cycles each of length $\frac{v-1}{k}$. $k$-rotational $3CS(v)$s are explored in [3, 19] in which the spectrum is determined for $k = 1, 2, 3, 4, 6$. An $n$-cycle system is said to be reverse if it admits an automorphism consisting of a fixed point and $\frac{v-1}{2}$ transpositions. A reverse $3CS(v)$ exists if and only if $v \equiv 1, 3, 9$ or $19 \pmod{24}$ [8, 22, 24, 25]. In this section, we give necessary and sufficient conditions for the existence of $k$-rotational $nCS(v)$s and reverse $nCS(v)$s for all $k$ and for $n = 4, 6, 8$. 73
We consider $k$-rotational $nCS(v)$s on the point set $\{\infty\} \cup Z_N \times Z_k$, where $N = \frac{v-1}{k}$, and with automorphism $\pi_k = (\infty)(0_0, 1_0, \ldots, (N-1)_0) \cdots (0_{k-1}, 1_{k-1}, \ldots, (N-1)_{k-1})$, where we represent the ordered pair $(x, y) \in Z_N \times Z_k$ as $x_y$.

We will represent the 4-cycle

```
   a
   |
   |
   |
   b
   |
   |
   |
   |
   |
   |
   c
   |
   |
   |
   d
```

by any cyclic shift of $(a, b, c, d)$ or $(d, c, b, a)$. We have the following necessary condition:

**Lemma 4.1** If a $k$-rotational $4CS(v)$ exists, then $k$ is even.

**Proof.** Suppose there is a $k$-rotational $4CS(v)$ with point set and automorphism as described above. A set of $n-$cycles is said to be a set of base $n-$cycles for an $nCS(v)$ under the automorphism $\pi$ if the images of the $n-$cycles under the powers of $\pi$ produce the $nCS(v)$. For each $i \in Z_k$, there must be exactly one 4-cycle in a set of base 4-cycles under the automorphism $\pi_k$ which contains the edge $(\infty, a_i)$, for some $a_i \in Z_N$. Base 4-cycles containing $\infty$ must be of one of the following types:

1. $(\infty, x_i, y_i, z_i)$ where $i \neq j$, or
2. $(\infty, x_i, y_j, z_m)$ where $i, j, m$ are distinct.

Each of these types of 4-cycles contains exactly two edges of the form $(\infty, a_i)$ where $a_i \in Z_N \times Z_k$. Therefore, $k$ must be even. 

Notice that the argument of Lemma 4.1 can be extended to show that if a $k$-rotational $nCS(v)$ exists where $n$ is even, then $k$ must be even.

We can now establish sufficiency.

**Lemma 4.2** A 2-rotational $4CS(v)$ exists if and only if $v \equiv 1 \pmod{8}$.

**Proof.** Suppose $v \equiv 1 \pmod{8}$, say $v = 8t + 1$. Let $N = 4t$. Consider the collection of 4-cycles:

- $(\infty, 0_0, 1_1, (2t)_1), (0_0, 1_0, (2t)_0, (3t)_0), (0_0, 0_1, (2t)_0, (2t)_1), (0_0, (3t)_1, 1_1, (2t)_0), (0_0, 1_1, (2t)_1, (2t + t)_0)$ for $i = 1, 2, \ldots, t - 1$ (omit if $t = 1$), and
- $(0_0, (4t - i)_1, (2t)_1, i_0)$ for $i = 1, 2, \ldots, t - 1$ (omit if $t = 1$).

These blocks, along with their images under $\pi_2$, form a 2-rotational $4CS(v)$.

The results of Lemmas 4.1 and 4.2 allow us to establish necessary and sufficient conditions for the existence of a $k$-rotational $4CS(v)$, for all $k$.

**Theorem 4.1** A $k$-rotational $4CS(v)$ exists if and only if $v \equiv 1 \pmod{k}$, $k$ is even and $v \equiv 1 \pmod{k}$.

**Proof.** In light of Lemma 4.1, the necessary conditions follow trivially. Now suppose $v$ and $k$ satisfy the stated hypotheses. Then there is a 2-rotational $4CS(v)$ admitting
automorphism $\pi_2$. Since $v \equiv 1 \pmod{k}$, we have $\pi_2^{k/2} = \pi_k$ and so the 2-rotational $4CS(v)$ is also $k$-rotational.

Theorem 4.1 immediately classifies reverse $4CS(v)$s:

**Corollary 4.1** A reverse $4CS(v)$ exists if and only if $v \equiv 1 \pmod{8}$.

We will represent the 6-cycle

![6-cycle](image)

by any cyclic shift of $(a, b, c, d, e, f)$ or $(f, e, d, c, b, a)$. We now consider $k$-rotational $6CS(v)$s. As with Lemma 4.1, a necessary condition for the existence of a $k$-rotational $6CS(v)$ is that $k$ is even.

We now establish sufficiency. In each of the constructions, we represent certain blocks with the following notation:

$A_j(a, i) = (0_j, (a + 6 + 12i)_j, (2a + 8 + 24i)_j, (3a + 8 + 36i)_j, (2a + 4 + 24i)_j, (a + 1 + 12i)_j),$

$B_j(b, i) = (0_j, (b + 6 + 12i)_j, (2b + 10 + 24i)_j, (3b + 10 + 36i)_j, (2b + 5 + 24i)_j, (b + 2 + 12i)_j),$

$C(c, i) = (0_j, (c + 6 + 12i)_j, 2_0, (c + 4 + 12i)_1, 1_0, (c + 1 + 12i)_1),$

$D(d, i) = (0_j, (d + 6 + 12i)_1, 1_0, (d + 5 + 12i)_1, 2_0, (d + 2 + 12i)_1).$

**Lemma 4.3** If $v \equiv 1 \pmod{48}$ then there exists a 2-rotational $6CS(v)$.

**Proof.** First, suppose that $v = 49$. Consider the collection of 6-cycles:

$(\infty, 3_0, 0_0, 2_1, 5_1, 11_1), (0_0, 6_0, 10_1, 22_1, 18_0, 12_0), (0_0, 11_1, 10_1, 8_0, 3_0, 9_1),$

$(0_0, 8_1, 3_0, 16_1, 16_0, 0_1), (0_0, 11_1, 10_1, 8_0, 3_0, 9_1), (12_0, (12_0), (17_0), (10_j, 4_j, 8_j, 12_j, 16_j, 20_j) \text{ for } j = 0, 1, \text{ and } B_j(5, 0) \text{ for } j = 0, 1.$

Now suppose $v \equiv 1 \pmod{48}$, say $v = 48t + 1$ where $t > 1$. Consider the collection of 6-cycles:

$(\infty, 3_0, 0_0, 2_1, 5_1, 9_1), (0_0, 4_0, 8_1, (12t + 4)_1, (12t + 4)_0, (12t)_0),$

$(0_0, (8t)_1, (8t)_0, (16t)_1, (16t)_0, 0_1), (0_0, 11_1, 10_1, 4_1, 10_0, 2_0),$

$(0_0, 4_0, 8_1, (12t)_1, (12t)_0, (16t)_1, (16t)_0, 20_0) \text{ for } j = 0, 1,$

along with

**Case 1.** If $t \equiv 0 \pmod{3}$, the 6-cycles

$A_j(5, i) \text{ for } i = 0, 1, \ldots, \frac{t-2}{3} \text{ for } j = 0, 1,$

$B_j(10, 0) \text{ for } i = 0, 1, \ldots, \frac{t-6}{3} \text{ for } j = 0, 1 \text{ (omitted if } t = 3),$

$(0_j, (4t + 4)_1, (8t + 6)_1, (12t + 7)_1, (8t + 1)_1, (4t - 2)_1) \text{ for } j = 0, 1,$

$B_j(4t + 5, i) \text{ for } i = 0, 1, \ldots, \frac{2t-3}{3} \text{ for } j = 0, 1,$

$A_j(4t + 12, i) \text{ for } i = 0, 1, \ldots, \frac{2t-6}{3} \text{ for } j = 0, 1,$

$C(5, i) \text{ for } i = 0, 1, \ldots, \frac{2t-3}{3}, D(10, 0) \text{ for } i = 0, 1, \ldots, \frac{2t-6}{3},$

$(0_0, (8t + 6)_1, 2_0, (8t + 5)_1, 4_0, (8t + 2)_1), D(8t + 5, i) \text{ for } i = 0, 1, \ldots, \frac{4t-3}{3}, \text{ and}$

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Case 2, if \( t \equiv 1 \pmod{3} \), \( t > 1 \), the 6-cycles

\[
A_j(5, i) \text{ for } i = 0, 1, \ldots, \frac{t-4}{3}; \\
B_j(5, 10, i) \text{ for } i = 0, 1, \ldots, \frac{t-7}{3} \text{ for } j = 0, 1 \text{ (omit if } t = 4) ; \\
(0, (4t + 2), (8t), (12t - 6), (8t - 7), (4t - 4)) \text{ for } j = 0, 1, \\
(0, (4t + 2), (8t + 3), (12t + 12), (8t + 7), (4t + 3)) \text{ for } j = 0, 1, \\
A_j((4t + 8, i) \text{ for } i = 0, 1, \ldots, \frac{2t-5}{3} \text{ for } j = 0, 1, \\
B_j((4t + 13, i) \text{ for } i = 0, 1, \ldots, \frac{2t-8}{3} \text{ for } j = 0, 1, \\
C((5, i) \text{ for } i = 0, 1, \ldots, \frac{2t-8}{3}, D((10, i) \text{ for } i = 0, 1, \ldots, \frac{2t-5}{3}, \\
(0, (8t + 3), (8t + 3), (8t), (8t + 3), (8t + 1)) \text{ for } i = 0, 1, \ldots, \frac{4t-4}{3}, \text{ and } \\
D((8t + 9, i) \text{ for } i = 0, 1, \ldots, \frac{4t-4}{3}.
\]

Case 3, if \( t \equiv 2 \pmod{3} \), the 6-cycles

\[
A_j(5, i) \text{ for } i = 0, 1, \ldots, \frac{t-5}{3} \text{ for } j = 0, 1 \text{ (omit if } t = 2) ; \\
B_j(5, 10, i) \text{ for } i = 0, 1, \ldots, \frac{t-5}{3} \text{ for } j = 0, 1 \text{ (omit if } t = 2) ; \\
(0, (4t + 3), (8t + 2), (12t - 2), (8t - 2), (4t - 3)) \text{ for } j = 0, 1, \\
A_j((4t + 4, i) \text{ for } i = 0, 1, \ldots, \frac{2t-4}{3} \text{ for } j = 0, 1, \\
B_j((4t + 9, i) \text{ for } i = 0, 1, \ldots, \frac{2t-4}{3} \text{ for } j = 0, 1, \\
C((5, i) \text{ for } i = 0, 1, \ldots, \frac{2t-4}{3}, D((10, i) \text{ for } i = 0, 1, \ldots, \frac{2t-7}{3} \text{ (omit if } t = 2), \\
(0, (8t + 2), (8t + 2), (8t), (8t + 2), (8t), (8t)) \text{ for } i = 0, 1, \ldots, \frac{4t-2}{3}, \text{ and } \\
C((8t + 8, i) \text{ for } i = 0, 1, \ldots, \frac{4t-5}{3}.
\]

In each case, the blocks, along with their images under \( \pi_2 \), form a 2-rotational \( 6CS(v) \).

Lemma 4.4 If \( v \equiv 9 \pmod{24} \) then there exists a 2-rotational \( 6CS(v) \).

Proof. First, suppose that \( v = 9 \). Consider the collection of 6-cycles:

\((\infty, 0, 0, 1, 1, 3, 2)\) and \((0, 1, 0, 2, 1, 0, 3, 0, 2)\).

Now, suppose \( v \equiv 9 \pmod{24} \), say \( v = 24t + 9 \) where \( t > 0 \). Consider the collection of 6-cycles:

\((\infty, 2, 0, 0, 3, 4, 8, 1)\), \((0, 0, 0, 3, 4, 6, 1, 3, 1)\), \((0, 0, 9, 15, 7, 17, 6, 0)\), \\
\((0, 4, 4, 6, 6, 1, 6, 6, 0)\), \((0, 5, 10, 1, 3, 2, 0)\), \((0, 5, 1, 1, 7, 1, 6, 0)\), \\
\((0, 5, 1, 2, 1, 3, 5, 0)\), \\
\(C(i, 6, i) \text{ for } i = 0, 1, \ldots, t - 2 \text{ (omit if } t = 1)\), \\
\(D(i, 21, i) \text{ for } i = 0, 0, 1, \ldots, t - 2 \text{ (omit if } t = 1)\),
along with

Case 1, if \( t \equiv 1 \pmod{2} \), the 6-cycles

\((0, 1, 3, 1, 20, 1, 3, 7, 0)\), \(A_j(8, i) \text{ for } i = 0, 1, \ldots, \frac{t-3}{2} \text{ for } j = 0, 1 \text{ (omit if } t = 1)\), and \\
\(B_j(13, i) \text{ for } i = 0, 1, \ldots, \frac{t-3}{2} \text{ for } j = 0, 1 \text{ (omit if } t = 1)\);

Case 2, if \( t \equiv 0 \pmod{2} \), \( t > 0 \), the 6-cycles

\((0, 12, 20, 1, 7, 22, 8)\), \(B_j(7, i) \text{ for } i = 0, 1, \ldots, \frac{t-2}{2} \text{ for } j = 0, 1, \text{ and } \\
A_j(14, i) \text{ for } i = 0, 1, \ldots, \frac{t-4}{2} \text{ for } j = 0, 1 \text{ (omit if } t = 2)\).

In each case, the blocks, along with their images under \( \pi_2 \), form a 2-rotational \( 6CS(v) \).

Lemma 4.5 If \( v \equiv 13 \pmod{48} \) then there exists a 2-rotational \( 6CS(v) \).

Proof. First, suppose that \( v = 13 \). Consider the collection of blocks:

(\text{Details of collection of 6-cycles for } v \equiv 13 \pmod{48})

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Next, suppose that \( v = 61 \). Consider the collection of blocks:
\[(\infty, 0_0, 2_0, 0_1, 1_1, 3_1), (0_0, 1_0, 4_1, 1_1, 4_0, 3_0), (0_0, 1_1, 2_0, 3_1, 4_0, 5_1), \text{ and } (0_0, 2_1, 2_0, 4_1, 4_0, 0_1).\]

Now, suppose \( v \equiv 13 \pmod{48} \), say \( v = 48t + 13 \) where \( t > 1 \). Consider the collection of 6-cycles:
\[(\infty, 3_0, 0_0, 2_1, 5_1, 9_1), (0_0, 4_0, 2_1, 3_6, 19_0, 15_0), (0_0, 5_1, 11_1, 4_0, 10_1, 6_0), (0_0, 1_1, 2_1, 4_1, 1_0, 2_0), (0_0, 8_0 + 2_1, 16_0 + 4_1, 16_0 + 4_0, 0_1), (0_0, 9_1, 16_1, 4_0, 15_1, 7_0), (0_0, 15_0, 17_0, 20_0, 25_0) \text{ for } j = 0, 1, B_j(8, 0) \text{ for } j = 0, 1, (0_0, 14_1, 23_1, 7_0, 22_1, 9_0), C(18, 0) \text{ for } j = 0, 1, \text{ and } (D(23, 0) \text{ for } j = 0, 1.\]

Along with

Case 1. If \( t \equiv 0 \pmod{3} \), \( t > 0 \), the 6-cycles
\[A_j(8, i) \text{ for } i = 0, 1, \ldots, \frac{t-4}{3} \text{ for } j = 0, 1,\]
\[B_j(13, i) \text{ for } i = 0, 1, \ldots, \frac{t-8}{3} \text{ for } j = 0, 1 \text{ (omitted if } t = 3),\]
\[A_j(4t + 3, i) \text{ for } i = 0, 1, \ldots, \frac{2t-3}{3} \text{ for } j = 0, 1,\]
\[B_j(4t + 8, i) \text{ for } i = 0, 1, \ldots, \frac{2t-3}{3} \text{ for } j = 0, 1, D(16, i) \text{ for } i = 0, 1, \ldots, \frac{2t-6}{3},\]
\[C(23, i) \text{ for } i = 0, 1, \ldots, \frac{2t-5}{3} \text{ (omitted if } t = 3), (0_0, 8_0 + 5_1, 1_0, 8_0 + 2_1, 3_0, 8_0 + 3_1),\]
\[C(8t + 6, i) \text{ for } i = 0, 1, \ldots, \frac{4t-3}{3}, \text{ and } D(8t + 11, i) \text{ for } i = 0, 1, \ldots, \frac{4t-3}{3}.\]

Case 2. If \( t \equiv 1 \pmod{3} \), \( t > 1 \), the 6-cycles
\[A_j(8, i) \text{ for } i = 0, 1, \ldots, \frac{t-4}{3} \text{ for } j = 0, 1,\]
\[B_j(13, i) \text{ for } i = 0, 1, \ldots, \frac{t-7}{3} \text{ for } j = 0, 1 \text{ (omitted if } t = 4),\]
\[(0_0, 4t + 7_1, 8t + 7_1, 12t + 4_1, 8t + 1_1, (4t - 1)_j) \text{ for } j = 0, 1,\]
\[(0_0, 4t + 10, 8t + 16, 12t + 21, 8t + 12, 4t + 4)_j \text{ for } j = 0, 1,\]
\[A_j(4t + 11, i) \text{ for } i = 0, 1, \ldots, \frac{2t-5}{3} \text{ for } j = 0, 1,\]
\[B_j(4t + 16, i) \text{ for } i = 0, 1, \ldots, \frac{2t-5}{3} \text{ for } j = 0, 1,\]
\[D(16, i) \text{ for } i = 0, 1, \ldots, \frac{2t-8}{3}, C(23, i) \text{ for } i = 0, 1, \ldots, \frac{2t-8}{3},\]
\[(0_0, 8t + 4_1, 3_0, 8t + 2_1, 2_0, 8t - 2_1), D(8t + 3, i) \text{ for } i = 0, 1, \ldots, \frac{4t-1}{3}, \text{ and }\]
\[C(8t + 10, i) \text{ for } i = 0, 1, \ldots, \frac{4t-4}{3}.\]

Case 3. If \( t \equiv 2 \pmod{3} \), the 6-cycles
\[A_j(8, i) \text{ for } i = 0, 1, \ldots, \frac{t-5}{3} \text{ for } j = 0, 1 \text{ (omitted if } t = 2),\]
\[B_j(13, i) \text{ for } i = 0, 1, \ldots, \frac{t-5}{3} \text{ for } j = 0, 1 \text{ (omitted if } t = 2),\]
\[B_j(4t, i) \text{ for } i = 0, 1, \ldots, \frac{2t-1}{3} \text{ for } j = 0, 1,\]
\[A_j(4t + 7, i) \text{ for } i = 0, 1, \ldots, \frac{2t-4}{3} \text{ for } j = 0, 1,\]
\[D(16, i) \text{ for } i = 0, 1, \ldots, \frac{2t-7}{3} \text{ (omitted if } t = 2),\]
\[C(23, i) \text{ for } i = 0, 1, \ldots, \frac{2t-5}{3} \text{ (omitted if } t = 2), (0_0, 8t + 8_1, 2_0, 8t + 7_1, 3_0, 8t + 3_1),\]
\[D(8t + 7, i) \text{ for } i = 0, 1, \ldots, \frac{4t-2}{3}, \text{ and } C(8t + 14, i) \text{ for } i = 0, 1, \ldots, \frac{4t-3}{3}.\]

In each case, the blocks, along with their images under \( \pi_2 \), form a 2-rotational 6CS(v).

Lemma 4.6 If \( v \equiv 21 \pmod{24} \) then there exists a 2-rotational 6CS(v).

Proof. Suppose \( v \equiv 21 \pmod{24} \), say \( v = 24k + 21 \). Consider the collection of 6-cycles:
\[(0_0, 3_0, 2_0, 6_1, 4_1, 1_1), D(3, i) \text{ for } i = 0, 1, \ldots, t,\]

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\[ C(10, i) \text{ for } i = 0, 1, \ldots, t - 1 \] (omit if \( t = 0 \)),

along with

**Case 1**, if \( t \equiv 0 \pmod{2} \), the 6-cycles

\((\infty, 2_0, 0_0, 2_1, 3_1, 7_1), (0_0, 4_0, 4_1, (6t + 9)_1, (6t + 9)_0, (6t + 5)_0),\)

\(A_j(5, i) \text{ for } i = 0, 1, \ldots, \frac{t-2}{2} \text{ for } j = 0, 1 \) (omit if \( t = 0 \)), and

\(B_j(10, i) \text{ for } i = 0, 1, \ldots, \frac{t-2}{2} \text{ for } j = 0, 1 \) (omit of \( t = 0 \));

**Case 2**, if \( t \equiv 1 \pmod{2} \), the 6-cycles

\((\infty, 2_0, 0_0, 2_1, 3_1, 8_1), (0_0, 5_0, 5_1, (6t + 10)_1, (6t + 10)_0, (6t + 5)_0),\)

\((0_1, 4_1, 12_1, 22_1, 16_1, 9_1) \text{ for } j = 0, 1,\)

\(A_j(11, i) \text{ for } i = 0, 1, \ldots, \frac{t-2}{3} \text{ for } j = 0, 1 \) (omit if \( t = 1 \)), and

\(B_j(16, i) \text{ for } i = 0, 1, \ldots, \frac{t-2}{3} \text{ for } j = 0, 1 \) (omit if \( t = 1 \)).

In each case, the blocks, along with their images under \( \pi_2 \), form a 2-rotational 6CS\( (v) \).

**Lemma 4.7** If \( v \equiv 25 \pmod{48} \) then there exists a 2-rotational 6CS\( (v) \).

**Proof.** First suppose that \( v = 25 \). Consider the collection of blocks:

\((\infty, 0_0, 5_0, 11_1, 2_1, 7_1), (0_0, 3_0, 5_0, 11_1, 0_0, 6_0), (0_0, 1_1, 2_1, 6_1, 3_0, 4_0),\)

\((0_1, 4_1, 4_0, 8_0, 0_0, 0_1), D(5, 0), (0_1, 2_1, 4_1, 5_1, 8_0, 10_0) \text{ for } j = 0, 1,\)

Now suppose \( v \equiv 25 \pmod{48} \), say \( v = 48t + 25 \) where \( t > 0 \). Consider the collection of 6-cycles:

\((\infty, 3_0, 0_0, 2_1, 5_1, 10_1), (0_0, 5_0, 9_1, (12t + 15)_1, (12t + 11)_0, (12t + 6)_0),\)

\((0_0, 1_1, 2_1, 4_0, 1_0, 2_0), (0_0, (8t + 4)_1, (8t + 4)_0, (16t + 8)_1, (16t + 8)_0, 0_1),\)

\((0_1, (4t + 2)_1, (8t + 4)_1, (12t + 6)_2, (16t + 8)_1, (20t + 10)_1) \text{ for } j = 0, 1,\)

along with

**Case 1**, if \( t \equiv 0 \pmod{3} \), \( t > 0 \), the 6-cycles

\(B_j(4, i) \text{ for } i = 0, 1, \ldots, \frac{t-4}{3} \text{ for } j = 0, 1,\)

\(A_j(11, i) \text{ for } i = 0, 1, \ldots, \frac{t-3}{4} \text{ for } j = 0, 1 \) (omit if \( t = 3 \)),

\((0_1, (4t + 5)_1, (8t + 6)_2, (12t + 6)_j, (8t + 2)_j, (4t - 1)_j) \text{ for } j = 0, 1,\)

\(A_j(4t + 6, i) \text{ for } i = 0, 1, \ldots, \frac{t-4}{3} \text{ for } j = 0, 1,\)

\(B_j(4t + 11, i) \text{ for } i = 0, 1, \ldots, \frac{4t-3}{3} \text{ for } j = 0, 1,\)

\(C(5, i) \text{ for } i = 0, 1, \ldots, \frac{2t-3}{3}, D(10, i) \text{ for } i = 0, 1, \ldots, \frac{2t-2}{3}\),

\(D(8t + 5, i) \text{ for } i = 0, 1, \ldots, \frac{4t-3}{3}, \) and \(C(8t + 12, i) \text{ for } i = 0, 1, \ldots, \frac{4t-3}{3}\);

**Case 2**, if \( t \equiv 1 \pmod{3} \), the 6-cycles

\(B_j(4, i) \text{ for } i = 0, 1, \ldots, \frac{t-4}{4} \text{ for } j = 0, 1 \) (omit if \( t = 1 \)),

\(A_j(11, i) \text{ for } i = 0, 1, \ldots, \frac{t-3}{4} \text{ for } j = 0, 1 \) (omit if \( t = 1 \)),

\((0_1, (4t + 8)_1, (8t + 13)_1, (12t + 13)_1, (8t + 7)_j, (4t + 3)_j) \text{ for } j = 0, 1,\)

\(B_j(4t + 7, i) \text{ for } i = 0, 1, \ldots, \frac{2t-2}{3} \text{ for } j = 0, 1,\)

\(A_j(4t + 14, i) \text{ for } i = 0, 1, \ldots, \frac{2t-5}{3} \text{ for } j = 0, 1 \) (omit if \( t = 1 \)),

\(C(5, i) \text{ for } i = 0, 1, \ldots, \frac{2t-2}{3}, D(10, i) \text{ for } i = 0, 1, \ldots, \frac{2t-5}{3} \) (omit if \( t = 1 \)),

\((0_0, (8t + 10)_1, 2_0, (8t + 9)_1, 3_0, (8t + 5)_1), D(8t + 9, i) \text{ for } i = 0, 1, \ldots, \frac{4t-1}{3}, \) and

\(C(8t + 16, i) \text{ for } i = 0, 1, \ldots, \frac{4t-3}{3};\)

**Case 3**, if \( t \equiv 2 \pmod{3} \), the 6-cycles

\(B_j(4, i) \text{ for } i = 0, 1, \ldots, \frac{t-5}{5} \text{ for } j = 0, 1 \) (omit if \( t = 2 \)),

\(A_j(11, i) \text{ for } i = 0, 1, \ldots, \frac{t-4}{5} \text{ for } j = 0, 1 \) (omit if \( t = 2 \)),

\((0_1, (4t + 4)_1, (8t + 3)_1, (12t - 1)_1, (8t - 2)_j, (4t - 2)_j) \text{ for } j = 0, 1,\)

\(B_j(4t + 3, i) \text{ for } i = 0, 1, \ldots, \frac{2t-1}{3} \text{ for } j = 0, 1,\)

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In each case, the blocks, along with their images under \( \pi_2 \), form a 2-rotational 6CS(v).

As in Theorem 4.1, the results of Lemmas 4.3 to 4.8 allow us to establish necessary and sufficient conditions for the existence of a \( k \)-rotational 6CS(v), for all \( k \).

**Theorem 4.2** A \( k \)-rotational 6CS(v) exists if and only if \( v \equiv 1 \) or \( 9 \pmod{12} \), \( k \) is even and \( v \equiv 1 \pmod{k} \).

Theorem 4.2 immediately classifies reverse 6CS(v):

**Corollary 4.2** A reverse 6CS(v) exists if and only if \( v \equiv 1 \) or \( 9 \pmod{12} \).

We will represent the 8-cycle

![Diagram of a 8-cycle diagram]

by any cyclic shift of \((a, b, c, d, e, f, g, h)\) or \((h, g, f, e, d, c, b, a)\). We now consider \( k \)-rotational 8CS(v)s. Again, as with Lemma 4.1, a necessary condition for the existence of a \( k \)-rotational 8CS(v) is that \( k \) is even.

We now establish sufficiency for \( k = 2 \) in a series of lemmas.

**Lemma 4.9** If \( v \equiv 1 \pmod{64} \) then there exists a 2-rotational 8CS(v).

**Proof.** First suppose that \( v = 65 \). Consider the collection of 8-cycles:

\[
(\infty, 0_0, 4_0, 16_0, 31_1, 15_0, 12_1, 24_1, 24_0, 16_0),
(0_1, 4_1, 8_1, 12_0, 16_1, 20_1, 24_0, 28_1),
(0_0, 12_1, 24_0, 4_1, 16_0, 28_1, 8_0, 20_1),
(0_1, 3_1, 7_1, 15_0, 14_1, 21_1, 2_0, 9_1) \] for \( i = 0, 1 \),

\[
(0_0, 2_0, 6_0, 16_0, 19_1, 14_0, 11_1, 13_1, 27_1),
(0_0, 26_1, 3_0, 28_1, 4_0, 10_1, 2_0, 9_1),
(0_0, 22_1, 4_0, 25_1, 6_0, 16_1, 3_0, 14_1).
\]

Now suppose \( v = 64t + 1 \) where \( t > 1 \). Consider the collection of 8-cycles:

\[
(\infty, 0_0, (4t)_0, (12t)_0, (28t - 1)_1, (12t - 1)_0, (28t)_1, (28t)_0, (16t)_0),
(0_0, (12t)_0, (12t)_1, 0_1, (16t)_0, (16t)_1, (28t)_1, (28t)_0, (16t)_0),
(0_1, (4t)_1, (8t)_1, (12t)_1, (16t)_0, (20t)_1, (24t)_1, (28t)_1),
(0_0, (4t)_1, (8t)_0, (12t)_0, (16t)_1, (20t)_1, (24t)_0, (28t)_1),
(0_0, (12t)_1, (24t)_0, (4t)_0, (16t)_0, (28t)_1, (8t)_0, (20t)_1),
(0_0, (16t)_0, (16t)_1, (16t - 1)_0, (16t - 2)_1, (16t - 1)_1, (32t - 2)_1),
(0_0, 2_0, (16t)_0, (16t)_1, (16t - 4)_1, (16t - 2)_1, (32t - 4)_1),
(0_1, (3 + 4s)_1, (7 + 8s)_1, (12 + 12s)_1, (18 + 16s)_1, (16t + 15 + 12s)_1, (11 + 8s)_1, (16t + 6 + 4s)_1) \] for \( s = 0, 1, \ldots, t - 2 \) and \( i = 0, 1 \),

\[
(0_1, (4t - 1)_t, (8t)_t, (12t + 2)_t, (16t + 5)_t, (28t + 6)_t, (8t + 5)_t, (20t + 3)_t) \] for \( i = 0, 1, \ldots, t - 2 \).
\((0, 4t + 4 + 4s)_i, (8t + 9 + 8s)_i, (12t + 15 + 12s)_i, (16t + 22 + 16s)_i, (28t + 18 + 12s)_i, (28t + 18 + 12s)_i, (8t + 13 + 8s)_i, (20t + 7 + 4s)_i)\) for \(s = 0, 1, \ldots, t - 2\) and \(i = 0, 1,\)
\((0, 0, (32t - 5 - 4s)_i, 3_0, (32t - 3 - 4s)_i, 4_0, (9 + 4s)_i, 2_0, (8 + 4s)_i)\) for \(s = 0, 1, \ldots, t - 3\) (omit if \(t = 2\)),
\((0, 0, (28t + 3)_i, 4_0, (28t + 6)_i, 5_0, (4t + 2)_i, 3_0, (4t + 1)_i)\),
\((0, 0, (28t - 2 - 4s)_i, 3_0, (28t - 4s)_i, 4_0, (4t + 6 + 4s)_i, 2_0, (4t + 5 + 4s)_i)\) for \(s = 0, 1, \ldots, 2t - 2\),
\((0, 0, (20t + 2)_i, 4_0, (20t + 5)_i, 5_0, (12t + 4)_i, 3_0, (12t + 2)_i)\), and
\((0, 0, (20t - 3 - 4s)_i, 3_0, (20t - 1 - 4s)_i, 4_0, (12t + 7 + 4s)_i, 2_0, (12t + 6 + 4s)_i)\) for \(s = 0, 1, \ldots, t - 2\).

In both cases, the blocks, along with their images under \(\pi_2\), form a 2-rotational 8CS\((v)\).

Lemma 4.10 If \(v = 17 \mod 64\) then there exists a 2-rotational 8CS\((v)\).

Proof. First suppose that \(v = 17\). Consider the collection of 8-cycles:
\((\infty, 0, 0, 2_0, 5_0, 1_1, 3_0, 1_3, 6_1), (0_0, 1_0, 3_1, 1_1, 5_1, 7_1, 5_0, 4_0),\)
\((0_1, 1_1, 2_1, 3_1, 4_1, 5_1, 6_1, 7_1), (0_0, 1_1, 2_0, 3_1, 4_0, 5_1, 6_0, 7_1),\) and
\((0_0, 3_0, 6_0, 1_1, 4_0, 7_1, 2_0, 5_1).\)

Now suppose \(v = 64t + 17\) where \(t > 0\). Consider the collection of 8-cycles:
\((\infty, 0_0, (4t + 1)_0, (12t + 3)_0, (28t + 8)_1, (12t + 4)_0, (28t + 7)_1, (20t + 5)_1),\)
\((0_0, (12t + 3)_0, (12t + 3)_1, 0_1, (16t + 4)_1, (28t + 7)_1, (28t + 7)_0, (16t + 4)_0),\)
\((0_1, (4t + 1)_1, (8t + 2)_1, (12t + 3)_1, (16t + 4)_1, (20t + 5)_1, (24t + 6)_1, (28t + 7)_1),\)
\((0_0, (4t + 1)_1, (8t + 2)_0, (12t + 3)_1, (16t + 4)_0, (20t + 5)_1, (24t + 6)_0, (28t + 7)_1),\)
\((0_0, (12t + 3)_1, (24t + 6)_0, (4t + 1)_1, (16t + 4)_0, (28t + 7)_1, (8t + 2)_0, (20t + 5)_1),\)
\((0_1, (1 + 4s)_1, (3 + 8s)_i, (6 + 12s)_i, (10 + 16s)_i, (16 + 13 + 12s)_i, (7 + 8s)_i, (16t + 8 + 4s)_i)\) for \(s = 0, 1, \ldots, t - 1\) and \(i = 0, 1,\)
\((0_1, (4t + 2 + 4s)_i, (8t + 5 + 8s)_i, (12t + 9 + 12s)_i, (16t + 14 + 16s)_i, (28t + 16 + 12s)_i, (8t + 9 + 8s)_i, (20t + 9 + 4s)_i)\) for \(s = 0, 1, \ldots, t - 1\) and \(i = 0, 1,\)
\((0_0, (32t + 7 - 4s)_1, 3_0, (1 - 4s)_1, 4_0, (5 + 4s)_1, 2_0, (4 + 4s)_1)\) for \(s = 0, 1, \ldots, t - 1,\)
\((0_0, (28t + 6 - 4s)_1, 3_0, (28t + 8 - 4s)_1, 4_0, (4t + 6 + 4s)_1, 2_0, (4t + 5 + 4s)_i)\) for \(s = 0, 1, \ldots, 2t - 1,\)
\((0_0, (20t + 6)_1, 4_0, (20t + 8)_1, 5_0, (12t + 7)_1, 2_0, (12t + 6)_1),\) and
\((0_0, (20t + 1 - 4s)_1, 3_0, (20t + 3 - 4s)_1, 4_0, (12t + 11 + 4s)_1, 2_0, (12t + 10 + 4s)_1)\) for \(s = 0, 1, \ldots, t - 2\) (omit if \(t = 1\)).

In both cases, the blocks, along with their images under \(\pi_2\), form a 2-rotational 8CS\((v)\).

Lemma 4.11 If \(v = 33 \mod 64\) then there exists a 2-rotational 8CS\((v)\).

Proof. First suppose that \(v = 33\). Consider the collection of 8-cycles:
\((\infty, 0_0, 2_0, 6_0, 13_1, 5_0, 14_1, 2_1), (0_0, 6_0, 6_1, 0_1, 8_1, 14_1, 14_0, 8_0),\)
\((0_1, 2_1, 4_1, 6_1, 8_1, 10_1, 12_1, 14_1), (0_0, 2_1, 4_0, 6_1, 8_0, 10_1, 12_0, 14_1),\)
\((0_0, 6_1, 12_0, 2_1, 8_0, 14_1, 4_0, 10_1), (0_0, 1_0, 8_0, 9_1, 6_0, 5_1, 6_1, 13_1),\) and
\((0_0, 3_0, 8_0, 12_1, 7_0, 3_1, 6_1, 11_1).\)

Now suppose \(v = 64t + 33\) where \(t > 0\). Consider the collection of 8-cycles:
\((\infty, 0_0, (4t + 2)_0, (12t + 6)_0, (28t + 13)_1, (12t + 5)_0, (28t + 14)_1, (20t + 10)_1),\)

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In both cases, the blocks, along with their images under \( \pi_2 \), form a 2-rotational 8CS(v).

Lemma 4.12 If \( v \equiv 49 \pmod{64} \) then there exists a 2-rotational 8CS(v).

Proof. Suppose \( v = 64t + 49 \) where \( t \geq 0 \). Consider the collection of 8-cycles:

\[
\begin{align*}
& (0_0, (12t + 6)_0, (12t + 6)_1, 0_1, (16t + 8)_1, (28t + 14)_1, (28t + 14)_0, (16t + 8)_0, \\
& (0_0, (4t + 2)_0, (8t + 4)_1, (12t + 6)_1, (16t + 8)_1, (20t + 10)_1, (24t + 12)_0, (28t + 14)_1, \\
& (0_0, (4t + 2)_1, (8t + 4)_0, (12t + 6)_1, (16t + 8)_0, (20t + 10)_1, (24t + 12)_0, (28t + 14)_1, \\
& (0_0, (12t + 6)_1, (24t + 12)_0, (4t + 2)_1, (16t + 8)_0, (28t + 14)_1, (8t + 4)_0, (20t + 10)_1, \\
& (0_0, 0_0, (16t + 8)_0, (16t + 9)_1, (16t + 7)_0, (16t + 6)_1, (16t + 7)_1, (32t + 14)_1, \\
& (0_0, 0_0, (16t + 8)_0, (16t + 11)_1, (16t + 7)_0, (16t + 4)_1, (16t + 5)_1, (32t + 12)_1, \\
& (0_0, (3 + 4s)_i, (7 + 8s)_i, (12 + 12s)_i, (18 + 16s)_i, (16t + 23 + 12s)_i, (11 + 8s)_i, (16t + 14 + 4s)_i \\
& \text{for } s = 0, 1, \ldots, t - 2 \text{ and } i = 0, 1 \text{ (omit if } t = 1), \\
& (0_0, (4t - 1)_i, (8t - 1)_i, (12t)_i, (16t + 3)_i, (28t + 12)_i, (8t + 4)_i, (20t + 11)_i \text{ for } i = 0, 1, \\
& (0_0, (4t + 4 + 4s)_i, (8t + 9 + 8s)_i, (12t + 15 + 12s)_i, (16t + 22 + 16s)_i, (28t + 26 + 12s)_i, \\
& (16t + 15 + 12s)_i, (32t + 14)_i, (8t + 9 + 8s)_i, (20t + 11 + 4s)_i \text{ for } s = 0, 1, \ldots, t - 1 \text{ and } i = 0, 1, \\
& (0_0, (32t + 11 - 4s)_1, 3_0, (32t + 13 - 4s)_1, 4_0, (9 + 4s)_i, 2_0, (8 + 4s)_1 \text{ for } s = 0, 1, \ldots, t - 2 \\
& \text{ (omit if } t = 1), \\
& (0_0, (28t + 15)_1, 4_0, (28t + 17)_1, 5_0, (4t + 6)_1, 2_0, (4t + 5)_1, \\
& (0_0, (28t + 10 - 4s)_1, 3_0, (28t + 12 - 4s)_1, 4_0, (4t + 10 + 4s)_i, 2_0, (4t + 9 + 4s)_1 \text{ for } s = 0, 1, \ldots, 2t - 1, \text{ and} \\
& (0_0, (20t + 9 - 4s)_1, 3_0, (20t + 11 - 4s)_1, 4_0, (12t + 11 + 4s)_i, 2_0, (12t + 10 + 4s)_1 \text{ for } s = 0, 1, \ldots, t - 1.
\end{align*}
\]

As in Theorem 4.1, the results of Lemmas 4.9 to 4.12 allow us to establish necessary
and sufficient conditions for the existence of a $k$-rotational $8CS(v)$, for all $k$.

**Theorem 4.3** A $k$-rotational $8CS(v)$ exists if and only if $v \equiv 1 \pmod{16}$, $k$ is even and $v \equiv 1 \pmod{k}$.

Theorem 4.3 immediately classifies reverse $8CS(v)$s:

**Corollary 4.3** A reverse $8CS(v)$ exists if and only if $v \equiv 1 \pmod{16}$.

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**References**


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