Decompositions of the Complete Digraph into each of the Orientations of a 4-Cycle which admit a Certain Automorphism

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Abstract. A decomposition of the complete digraph on \( v \) vertices, \( D_v \), is said to be \( f \)-cyclic if it admits an automorphism consisting of \( f \) fixed points and a single cycle of length \( v - f \). Necessary and sufficient conditions are given for the existence of \( f \)-cyclic decompositions of the complete digraph into each of the four orientations of a 4-cycle.

1 Introduction

Let \( D_v \) denote the complete digraph on \( v \) vertices. If \( g \) is a digraph, then a \( g \)-decomposition of \( D_v \) is a set \( \gamma = \{g_1, g_2, \ldots, g_n\} \) of arc-disjoint subgraphs of \( D_v \) such that each \( g_i \) (which is called a block of the decomposition) is isomorphic to \( g \) and \( \bigcup_{i=1}^{n} A(g_i) = A(D_v) \), where \( A(G) \) is the arc set of digraph \( G \). An automorphism of a \( g \)-decomposition of \( D_v \) is a permutation of the vertex set of \( D_v \) which fixes the set \( \gamma \).

There are two orientations of the 3-cycle: the 3-circuit and the following digraph (called a "transitive triple"):

```
  c
 /|
/  |
 a   b
```

A decomposition of \( D_v \) into 3-circuits is equivalent to a Mendelsohn triple system of order \( v \) denoted \( MTS(v) \) [9]. A decomposition of \( D_v \) into transitive triples is equivalent to a directed triple system of order \( v \), denoted \( DTS(v) \) [8].

There are four orientations of the 4-cycle: the 4-circuit and the following:

```
 b   c
 a   d
    X
```
```
 b   c
 a   d
    Y
```
```
 b   c
 a   d
    Z
```

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We represent $X$ as $[a, b, c, d]_X$, $Y$ as $[a, b, c, d]_Y$, and $Z$ as $[a, b, c, d]_Z$. We represent the 4-circuit with arc set $\{(a, b), (b, c), (c, d), (d, a)\}$ by any cyclic shift of $[a, b, c, d]_c$. A 4-circuit decomposition of $D_v$ exists if and only if $v \equiv 0$ or 1 (mod 4), $v \neq 4$ [14]. An $X$-decomposition of $D_v$ exists if and only if $v \equiv 0$ or 1 (mod 4), $v \neq 5$; a $Y$-decomposition of $D_v$ exists if and only if $v \equiv 0$ or 1 (mod 4), $v \notin \{4, 5\}$; and a $Z$-decomposition of $D_v$ exists if and only if $v \equiv 1$ (mod 4) [6].

A digraph decomposition admitting an automorphism consisting of a single cycle is said to be cyclic. A cyclic $MTS(v)$ exists if and only if $v \equiv 1$ or 3 (mod 6), $v \neq 9$ [4] and a cyclic $DTS(v)$ exists if and only if $v \equiv 1$, 4, or 7 (mod 12) [5]. A cyclic 4-circuit decomposition of $D_v$ exists if and only if $v \equiv 1$ (mod 4) [11]; a cyclic $X$-decomposition of $D_v$ exists if and only if $v \equiv 1$ (mod 4), $v \neq 5$; a cyclic $Y$-decomposition of $D_v$ exists if and only if $v \equiv 1$ (mod 4), $v \neq 5$; and a cyclic $Z$-decomposition of $D_v$ exists if and only if $v \equiv 1$ (mod 4) [3,10].

A decomposition of $D_v$ admitting an automorphism consisting of a fixed point and a cycle of length $v - 1$ is said to be rotational. A rotational $MTS(v)$ exists if and only if $v \equiv 1, 3, or 4$ (mod 6), $v \neq 10$ [1], and a rotational $DTS(v)$ exists if and only if $v \equiv 0$ (mod 3) [2]. A rotational 4-circuit decomposition of $D_v$ exists if and only if $v \equiv 1$ (mod 4) [13]; a rotational $X$-decomposition of $D_v$ exists if and only if $v \equiv 0$ (mod 4); a rotational $Y$-decomposition of $D_v$ exists if and only if $v \equiv 0$ (mod 4), $v \neq 4$; and a rotational $Z$-decomposition of $D_v$ does not exist [3].

A decomposition of $D_v$ which admits an automorphism consisting of $f$ fixed points, $f > 1$, and a single cycle of length $v - f$ is said to be $f$-cyclic. Necessary and sufficient conditions for the existence of a $f$-cyclic $MTS(v)$ are given in [7] and for a $f$-cyclic $DTS(v)$ are given in [12]. The purpose of this paper is to give necessary and sufficient conditions for the existence of a $f$-cyclic $g$-decomposition of $D_v$ where $g$ is an orientation of the 4-cycle.

2 The Constructions

In this section we give necessary and sufficient conditions for the existence of a $g$-decomposition of $D_v$, where $g$ is an orientation of the 4-cycle, which admits an automorphism consisting of $f$ fixed points and a cycle of length $v - f$. Throughout this section we suppose the vertex set of $D_v$ is $\{0_0, 1_0, \ldots, (f - 1)_0, 0_1, 1_1, \ldots, (v - f - 1)_1\}$ and let the relevant automorphism be $(0_0)(1_0) \cdots ((f - 1)_0)(0_1, 1_1, \ldots, (v - f - 1)_1)$.

We need a preliminary result before presenting the constructions.

**Lemma 2.1** If $\pi$ is an automorphism of a $g$-decomposition of $D_v$, then the fixed points of $\pi$ form a sub-$g$-decomposition. That is, if $\pi(x_0) = x_0$ and $\pi(y_0) = y_0$ for $(x_0, y_0) \in A(g_0)$, then $\pi(g_0) = g_0$.

**Proof.** If $(x_0, y_0) \in A(g_0)$ then by the definition of automorphism, $(\pi(x_0), \pi(y_0)) \in A(\pi(g_0))$. But then $(x_0, y_0) \in A(\pi(g_0))$ and since $(x_0, y_0)$ is in the arc set of exactly one $g_i$, it must be that $g_0 = \pi(g_0)$.

We have a necessary condition for the existence of a $f$-cyclic 4-circuit decomposition of $D_v$: 34
Lemma 2.2 If \( v \equiv 0 \pmod{4} \) and \( v = f + 4 \), then a \( f \)-cyclic 4-circuit decomposition of \( D_v \) does not exist.

Proof. Suppose such a system does exist. From Lemma 2.1, it follows that arcs of type \((a_1, b_1)\) must be contained in blocks of the form \([w_0, x_1, y_1, z_1]_C\) or \([w_1, x_1, y_1, z_1]_C\). Now each set

\[
\{\pi^n([w_0, x_1, y_1, z_1]_C) \mid n \in \mathbb{Z}, w \in \mathbb{Z}_f, \{x, y, z\} \subset \mathbb{Z}_{u-f}\}
\]

is of cardinality 4 and so the total number of arcs of type \((a_1, b_1)\) in blocks of the form \([w_0, x_1, y_1, z_1]_C\) is a multiple of 8. Therefore, such a system can have at most one fixed point in blocks of this form, since under our hypotheses \( D_v \) contains only 8 arcs of type \((a_1, b_1)\). Therefore each remaining fixed point must be contained in some block of the form \([w_0, x_1, y_0, z_1]_C\) (since each arc of the form \((a_0, b_1)\) is contained in some block). However, such blocks contain two distinct fixed vertices. Therefore, the cardinality of the set

\[
\{w_0 \mid w_0 \in V([w_0, x_1, y_0, z_1]_C), \{w, y\} \subset \mathbb{Z}_f, \{x, z\} \subset \mathbb{Z}_{u-f}\}
\]

is even. This implies that the cardinality of the set

\[
\{w_0 \mid w_0 \in V([w_0, x_1, y_1, z_1]_C), w \in \mathbb{Z}_f, \{x, y, z\} \subset \mathbb{Z}_{u-f}\}
\]

is even. However as seen above, the cardinality of this set can be at most 1. Therefore, the cardinality of this set must be 0, and all arcs of the type \((a_1, b_1)\) must be contained in blocks of the form \([w_1, x_1, y_1, z_1]_C\). However, the only such admissible blocks are \([0_1, 1_1, 2_1, 3_1]_C\) and \([3_1, 2_1, 1_1, 0_1]_C\), both of which are fixed under \( \pi \) and both of which contain 4 arcs of the form \((a_1, b_1)\). Under our hypotheses, \( D_v \) contains 12 arcs of the form \((a_1, b_1)\), therefore such a system cannot exist.

Theorem 2.1 An \( f \)-cyclic 4-circuit decomposition of \( D_v \) exists if and only if \( f \equiv 0 \) or 1 \((\text{mod} 4)\), \( f \neq 4 \), \( v \equiv 0 \) or 1 \((\text{mod} 4)\), \( v \neq 4 \), and \( v-f \geq 8 \) in the case \( f = v = 0 \) \((\text{mod} 4)\).

Proof. The fact that a 4-circuit decomposition of \( D_v \) exists only if \( v = 0 \) or 1 \((\text{mod} 4)\), \( v \neq 4 \), along with Lemma 2.1, give the necessary congruence conditions on \( v \) and \( f \). The necessity of \( v \geq f+8 \) for \( f \equiv v \equiv 0 \) \((\text{mod} 4)\) is given in Lemma 2.2. These conditions are shown to be sufficient in the following four cases.

Case 1. Suppose \( f \equiv 0 \) \((\text{mod} 4)\), \( f \neq 4 \), \( v \equiv 0 \) \((\text{mod} 4)\), \( v \neq 4 \), and \( v \geq f+8 \). Say \( v-f = 4t \). Consider the blocks:

\[
[0_1, 1_1, (t+2i-1)_1, (t+i-1)_1]_C \text{ for } i = 2, 3, \ldots, t-1, \\
[(2i)_{0_1}, 0_1, (2i+1)_{0_1}, 1_1]_C \text{ for } i = 1, 2, \ldots, f/2-1. \\
[0_1, 1_1, (2i)_{0_1}, (2i+1)_1]_C, \quad [0_1, 1_1, (2i+1)_{0_1}, (2i+2)_1]_C, \\
[0_0, 1_1, (2t+1)_1, (t+1)_1]_C, \text{ and } [1_0, 0_1, (2t+1)_1, (2t)_1]_C.
\]

Case 2. Suppose \( v \equiv 0 \) \((\text{mod} 4)\), \( f \neq 4 \), \( v \equiv 1 \) \((\text{mod} 4)\) and \( v \geq f+8 \). Say \( v-f = 4t-1 \). Consider the blocks:

\[
[(2i)_{0_1}, 1_1, (2i+1)_0, 1_1]_C \text{ for } i = 0, 1, \ldots, f/2-1, \\
\text{ and the blocks for a cyclic 4-circuit decomposition of } D_{u-f} \text{ on the vertex set } \\
\{0_1, 1_1, \ldots, (v-f-1)_1\}.
\]
Case 3. Suppose \( f = 1 \pmod{4} \) and \( v = 0 \pmod{4} \), \( v \neq 4 \) and \( v \geq f + 8 \). Say \( v - f = 4t - 1 \). Consider the blocks:
\[
(0_i, 1 + 2i), (3 + 4i), (2 + 2i) \mod{C} \quad \text{for} \quad i = 0, 1, \ldots, t - 3;
\]
\[
[(3 + 2i)_{0i}, (4 + 2i)_{0i}, 1]_C \quad \text{for} \quad i = 0, 1, \ldots, (f - 5)/2;
\]
\[
[0, 0_1, (2t - 3)_{11}, (4t - 3)]_C \quad \text{for} \quad i = 0, 1, (2t - 1), (4t - 3)_C \quad \text{and} \quad [2, 0, 1, (2t + 1), 4]_C.
\]

Case 4. Suppose \( f = 1 \pmod{4} \) and \( v = 1 \pmod{4} \) and \( v \geq f + 8 \). Say \( v - f = 4t \).

Consider the blocks:
\[
(0_i, 1 + 2i), (t + 2i)_1, (t + i)_1 \mod{C} \quad \text{for} \quad i = 1, 2, \ldots, t - 1;
\]
\[
[(2i - 1)_0, 0_1, (2i)_0]_C \quad \text{for} \quad i = 1, 2, \ldots, (f - 1)/2;
\]
\[
[0, 1_i, (2i)_{11}, (3i)]_C \quad \text{and} \quad [0, 0, (2t - 1), (t - 1)]_C.
\]

In each case, these blocks, along with their images under the permutation \((0_0, 1_0, \ldots, (f - 1)_0, 0_1, 1_1, \ldots, (v - f - 1)_0)\) and the blocks for a 4-circuit decomposition of \( D_f \) on the vertex set \( \{0_0, 1_0, \ldots, (f - 1)_0\} \), form an \( f \)-cyclic 4-circuit decomposition of \( D_v \).

Lemma 2.3 An \( f \)-cyclic \( X \)-decomposition of \( D_v \) satisfies the condition \( v \geq 3f + 1 \).

Proof. First, we observe that it is impossible for such a decomposition to contain a block of the form \([w_0, x_1, y_0, z_1]_X\). Applying \( \pi^{x_0} \) yields \([\pi^{x_0}(w_0), \pi^{x_0}(x_1), \pi^{x_0}(y_0), \pi^{x_0}(z_1)]_X = [w_0, \pi^{x_0}(x_1), y_0, z_1]_X\), a contradiction since these are distinct blocks which both contain the arc \((w_0, x_1)\). Similarly, such a decomposition cannot contain blocks of the form \([w_1, x_0, y_1, z_0]_X\). Therefore by Lemma 2.1, for each fixed point \( w_0 \), we have \( w_0 \in V(g_{w_0}) \) for some \( g_{w_0} \) where \( V(g_{w_0}) = \{w_0, x_1, y_1, z_1\} \). Let \( S_{w_0} = \bigcup_{n \in \mathbb{Z}} A(\pi^n(g_{w_0})) \) and
\[
S = \bigcup_{\{w_0, x_0, y_0, z_0\} \in \{0_0, 1_0, \ldots, (f - 1)_0\}} S_{w_0}.
\]

Now, there are \((v - f)(v - f - 1)\) arcs of the form \((a_1, b_1)\) in \( A(D_v) \) and there are \(2f(v - f)\) arcs of this form in \( S \). So it is necessary that \((v - f)(v - f - 1) \geq 2f(v - f)\), or that \( v \geq 3f + 1 \).

Theorem 2.2 An \( f \)-cyclic \( X \)-decomposition of \( D_v \) exists if and only if \( v \geq 3f + 1 \) and either \( f = 0 \pmod{4} \) and \( v = 1 \pmod{4} \), \( v \neq 5 \), or \( f = 1 \pmod{4} \), \( f \neq 5 \), and \( v = 0 \pmod{4} \).

Proof. As seen in the proof of Lemma 2.3, each block of such a decomposition must be of one of the following forms: \([w_0, x_0, y_0, z_0]_X\), \([w_1, x_0, y_1, z_1]_X\), \([w_0, x_1, y_0, z_1]_X\), or \([w_1, x_1, y_1, z_1]_X\). Now, the cardinality of the sets \( \{\pi^n([w_1, x_0, y_1, z_1]_X) \mid n \in \mathbb{Z}\} \) \( \{\pi^n([w_1, x_0, y_1, z_1]_X) \mid n \in \mathbb{Z}\} \) and \( \{\pi^n([w_1, x_1, y_1, z_1]_X) \mid n \in \mathbb{Z}\} \) are each \((v - f)\). Since each of these blocks contains an even number of arcs of the type \((a_1, b_1)\), it must be that the total number of such arcs is an even multiple of \((v - f)\). However, there are \((v - f)(v - f - 1)\) arcs of this type in \( A(D_v) \), and so it is not possible that \( f = v \pmod{4} \). This condition, along with Lemmas 2.1 and 2.3 and the conditions for the existence of a \( X \)-decomposition of \( D_v \) gives the necessary conditions for the existence of a \( f \)-cyclic \( X \)-decomposition of \( D_v \). We now establish sufficiency in the following four cases:

Case 1. Suppose \( f = 1 \pmod{4}, f \neq 5, v = 0 \pmod{4}, v - f = 7 \pmod{8}, \) and
\( v \geq 3f + 1 \). Say \( v - f = 8t - 1 \). Consider the blocks:

\[ [0, (f - 1)/2, (f - 1)/2 + 1, \ldots, t - 1 \text{ (omit if } t < (f + 1)/2) \],
\[ [0, (f + 1)/2, (f - 1)/2 + t + 1, \ldots, t - 2 \text{ (omit if } 2t < (f + 3)/2) \],
\[ (f - 1)/2 + t + 1 \text{ (omit if } (f - 1)/2 - t < 1) \],
\[ (f - 1)/2 - t \text{ (omit if } (f - 1)/2 - t < 1) \],
\[ (6t - 1)/2, 0, 1, (6t - 2)/2]_X \].

**Case 2.** Suppose \( f \equiv 1 \text{ (mod 4), } v \equiv 0 \text{ (mod 4), } v - f \equiv 3 \text{ (mod 8), and } v \geq 3f + 1 \). Say \( v - f = 8t + 3 \). Consider the blocks:

\[ [0, (f - 1)/4, (f - 1)/4 + 1, \ldots, t - 1 \],
\[ [0, (f - 1)/4, (f - 1)/4 + t - 1 \text{ (omit if } (f - 1)/4 - 1 \)],
\[ (f - 1)/4 + 1 \text{ (omit if } (f - 1)/4 - 1 \)],
\[ (f - 1)/4 + t - 1 \text{ (omit if } (f - 1)/4 - 1 \)],
\[ (3f - 1)/4 + t - 1 \text{ (omit if } (f - 1)/4 - 1 \)],
\[ (6t - 1)/2, 0, 1, (6t - 2)/2]_X \].

**Case 3.** Suppose \( f \equiv 0 \text{ (mod 4), } v \equiv 1 \text{ (mod 4), } v - f \equiv 5 \text{ (mod 8), and } v \geq 3f + 1 \). Say \( v - f = 8t + 5 \). Consider the blocks:

\[ [0, f/4, f/4 + 1, \ldots, t - 1 \],
\[ [0, f/4, f/4 + t - 1 \text{ (omit if } f/4 - 1 \)],
\[ f/4 + t - 1 \text{ (omit if } f/4 - 1 \)],
\[ f/4 + t - 1 \text{ (omit if } f/4 - 1 \)],
\[ f/4 + t - 1 \text{ (omit if } f/4 - 1 \)],
\[ (6t - 1)/2, 0, 1, (6t - 2)/2]_X \].

**Case 4.** Suppose \( f \equiv 0 \text{ (mod 4), } v \equiv 1 \text{ (mod 4), } v - f \equiv 5 \text{ (mod 8), and } v \geq 3f + 1 \). Say \( v - f = 8t + 5 \). Consider the blocks:

\[ [0, (f - 1)/2, (f - 1)/2 + 1, \ldots, t \text{ (omit if } t < (f + 1)/2) \],
\[ [0, (f - 1)/2, (f - 1)/2 + t - 1 \text{ (omit if } f < (f + 1)/2) \],
\[ (f - 1)/2 + t - 1 \text{ (omit if } (f - 1)/2 - t < 1) \],
\[ (f - 1)/2 - t \text{ (omit if } (f - 1)/2 - t < 1) \],
\[ (6t - 1)/2, 0, 1, (6t - 2)/2]_X \].

In each case, these blocks, along with their images under the permutation \((0)_0(1)_0 \cdots (f - 1)_0(0)_0, 1, \ldots, v - f - 1)\) and the blocks for a \(X\)-decomposition of \(D_f\) on the vertex set \(\{0, 1, \ldots, (f - 1)_0\}\), form an \(f\)-cyclic \(X\)-decomposition of \(D_v\).

**Theorem 2.3** An \(f\)-cyclic \(Y\)-decomposition of \(D_v\) exists if and only if either \(f \equiv 0\)
(mod 4), \( f \neq 4 \), and \( v \equiv 1 \) (mod 4), \( v \neq 5 \) or \( f \equiv 1 \) (mod 4), \( f \neq 5 \), and \( v \equiv 0 \) (mod 4), \( v \neq 4 \).

**Proof.** By Lemma 2.1, each arc of the form \((a_1, b_1)\) must be contained in a block of one of the following forms: \([w_1, x_0, y_1, z_1]_\gamma\) or \([w_1, x_1, y_1, z_1]_\gamma\). Now, the cardinality of the sets \(\{\pi^n([w_1, x_0, y_1, z_1]_\gamma) \mid n \in \mathbb{Z}\}\) and \(\{\pi^n([w_1, x_1, y_1, z_1]_\gamma) \mid n \in \mathbb{Z}\}\) are both \((v-f)\). Since each of these blocks contains an even number of arcs of the form \((a_1, b_1)\), it must be that the total number of such arcs is an even multiple of \((v-f)\). However, there are \((v-f)\) arcs of this form in \(A(D_v)\), and so it is not possible that \(f \equiv v \) (mod 4). This condition, along with Lemma 2.1 and the conditions for the existence of a \(Y\)-decomposition of \(D_v\) gives the necessary conditions for the existence of an \(f\)-cyclic \(Y\)-decomposition of \(D_v\). We now establish sufficiency in the following two cases:

**Case 1.** Suppose \(f \equiv 1 \) (mod 4), \( f \neq 5 \), and \( v \equiv 0 \) (mod 4), \( v \neq 4 \). Then \( v-f \equiv 3 \) (mod 4), say \( v-f = 4t-1 \). Consider the blocks:

\[
[0_1, (1+i)_1, (4t-3)_1, (2t-1+i)_1]_\gamma \quad \text{for} \quad i = (f-1)/2, (f-1)/2 + 1, \ldots, t - 2,
\]

\[
[0_1, (1+i)_0, (4t-3)_1, (2t-1+i)_1]_\gamma \quad \text{for} \quad i = 0, 1, \ldots, (f-1)/2 - 1,
\]

\[
[0_1, ((f-1)/2+1+i)_0, (4t-3)_1, (1+i)_1]_\gamma \quad \text{for} \quad i = 0, 1, \ldots, (f-1)/2 - 1, \quad \text{and}
\]

\[
[1_1, 0_0, (4t-3)_1, 0_1]_\gamma.
\]

**Case 2.** Suppose \(f \equiv 0 \) (mod 4), \( f \neq 4 \), and \( v \equiv 1 \) (mod 4), \( v \neq 5 \). Then \( v-f \equiv 1 \) (mod 4), say \( v-f = 4t+1 \). Consider the blocks:

\[
[0_1, (1+i)_1, (4t-1)_1, (2t+1+i)_1]_\gamma \quad \text{for} \quad i = f/2 - 1, f/2, \ldots, t - 2,
\]

\[
[0_1, i_0, (4t-1)_1, (2t+1+i)_1]_\gamma \quad \text{for} \quad i = 0, 1, \ldots, f/2 - 2,
\]

\[
[0_1, (f/2+1+i)_0, (4t-1)_1, (1+i)_1]_\gamma \quad \text{for} \quad i = 0, 1, \ldots, f/2 - 2, \quad \text{and}
\]

\[
[0_1, (2t-1)_1, (2t-2)_1, (4t-1)_1]_\gamma \quad \text{and} \quad [0_1, (f-2)_0, 0_1, (f-1)_0]_\gamma.
\]

In either case, these blocks, along with their images under the permutation \((0_0)(1_0) \cdots (f-1)_0(0_1, 1_1, \ldots, (v-f-1)_1)\) and the blocks for a \(Y\)-decomposition of \(D_f\) on the vertex set \(\{0_0, 1_0, \ldots, (f-1)_0\}\), form an \(f\)-cyclic \(Y\)-decomposition of \(D_v\).

**Theorem 2.4** An \(f\)-cyclic \(Z\)-decomposition of \(D_v\) does not exist.

**Proof.** Suppose that such a system exists. We observe that the system can contain no blocks of the form \([u_0, x_1, y_1, z_1]_Z\) or \([x_1, u_0, y_1, z_1]_Z\), for applying \(\pi^{x-z}\) to such blocks leads to a contradiction, as in the proof of Lemma 2.3. So all arcs of the form \((a_1, b_1)\) must be contained in blocks of the form \([w_1, x_1, y_1, z_1]_Z\). Therefore, there is a cyclic subsystem of the given system of order \((v-f)\). So \(v-f \equiv 1 \) (mod 4). But by Lemma 2.1, \(f \equiv 1 \) (mod 4) and so \(v \equiv 2 \) (mod 4), a contradiction.

**References**


