Bicyclic Decompositions of $K_v$ into Copies of $K_3 \cup \{e\}$

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Abstract. A decomposition of the complete graph on $v$ vertices, $K_v$, into copies of $K_3$ with a pendant edge is called a "lollipop" system of order $v$, denoted $LS(v)$. We give necessary and sufficient conditions for the existence of a $LS(v)$ admitting an automorphism consisting of two disjoint cycles. We also give a brief proof that the previously known sufficient conditions for the existence of a cyclic $LS(v)$ are in fact necessary.

1 Introduction

A $G$-design on $H$ is a set $\{g_1, g_2, \ldots, g_n\}$ of subgraphs of $H$ (called blocks) such that $g_i \cong G$ for $i \in \{1, 2, \ldots, n\}$, $E(g_i) \cap E(g_j) = \emptyset$ for $i \neq j$, and $\bigcup_{i=1}^n E(g_i) = E(H)$. Notice that a $G$-design on $H$ is equivalent to a $G$-decomposition of $H$. Several $G$-designs on the complete graph, $K_v$, have been explored. In particular, necessary and sufficient conditions are known for such designs for $G \in \{K_3, S_n\} \bigcup \{C_n \mid n \leq 50\}$ where $S_n$ denotes a star on $n+1$ vertices and $C_n$ denotes a cycle on $n$ vertices (see, for example, [1, 5, 10]). We are particularly interested in $L$-designs of $K_v$ when $L$ is the following graph:

\[ L = \]

\[\begin{array}{c}
 a \\
 \downarrow \\
 c \\
 \downarrow \\
 b \\
 \downarrow \\
 d \\
\end{array}\]

We denote $L$ as given here by either $(a, b, c) - d$ or $(b, a, c) - d$. Bermond and Schönheim proved that an $L$-design on $K_v$ exists if and only if $v \equiv 0$ or 1

Utilitas Mathematica 54(1998), pp. 51-57
(mod 8) [2]. More generally, Hoffman and Kirkpatrick recently proved an L-design on \(\lambda K_v\) exists if and only if \(\lambda v(v - 1) \equiv 0 \pmod{8}\) [7]. Since the graph \(L\) is colloquially known as a "lollipop" [8], we refer to an \(L\)-design on \(K_v\) as a lollipop system of order \(v\), denoted \(LS(v)\).

An automorphism of a \(G\)-design on \(H\) is a permutation of \(V(H)\) which fixes the set \(\{g_1, g_2, \ldots, g_n\}\). Such an automorphism is said to be cyclic if it consists of a cycle of length \(|V(H)|\) and is said to be bicyclic if it consists of a cycle of length \(M\) and a cycle of length \(N\) where \(M + N = |V(H)|\).

A \(K_3\)-design of \(K_v\) is also known as a Steiner triple system of order \(v\), denoted \(STS(v)\). A cyclic \(STS(v)\) exists if and only if \(v \equiv 1\) or 3 (mod 6), \(v \neq 9\) [9] and necessary and sufficient conditions for the existence of a bicyclic \(STS(v)\) are given in [3].

Bermond and Schönheim took advantage of "difference methods" in showing the existence of \(LS(v)\)s and proved that a cyclic \(LS(v)\) exists if \(v \equiv 1 \pmod{8}\). We shall briefly show these conditions are, in fact, necessary. With the help of this result, we will then give necessary and sufficient conditions for the existence of a bicyclic \(LS(v)\).

2 Cyclic Designs

For cyclic \(LS(v)\)s, we assume that the vertex set of \(K_v\) is \(\{0, 1, \ldots, v - 1\}\) and that the cyclic automorphism is \(\pi_c = (0, 1, \ldots, v - 1)\).

**Theorem 2.1** A cyclic \(LS(v)\) exists if and only if \(v \equiv 1 \pmod{8}\).

**Proof.** Suppose there is a cyclic \(LS(v)\) where \(v \equiv 0 \pmod{8}\). There must be some \(g_i\) in such a design which contains the edge \((0, v/2)\). Applying \(\pi_c^{v/2}\) to \(g_i\), we see that \((0, v/2)\) is an edge of \(\pi_c^{v/2}(g_i)\) and therefore \(\pi_c^{v/2}(g_i) = g_i\). However, this is impossible. Therefore \(v \not\equiv 0 \pmod{8}\) and this, combined with the necessary condition for existence of a \(LS(v)\), gives the necessary condition. Sufficiency is given in [2].

3 Bicyclic Designs

Throughout this section, we assume the vertex set of \(K_v\) is \(\{0_0, 1_0, \ldots, (M - 1)_0, 0_1, 1_1, \ldots, (N - 1)_1\}\) where \(M + N = v\) and we will construct \(LS(v)\)s admitting \(\pi = (0_0, 1_0, \ldots, (N - 1)_0)(0_1, 1_1, \ldots, (M - 1)_1)\) as an automorphism. First we give necessary conditions for such a design.
Lemma 3.1 A bicyclic LS(v) admitting an automorphism consisting of a cycle of length M and a cycle of length N where M = N = v/2 does not exist.

Proof. Suppose there is such a system. There must be some \( g_i \) in such a design which contains the edge \((0_0, (v/4)_0)\). As in Theorem 2.1, \( \pi^{v/4}(g_i) = g_i \). Therefore edge \((0_0, (v/4)_0)\) must be in a copy of \( L \) of the form \((0_0, (v/4)_0, c) - d \) for some vertices \( c \) and \( d \). But then we need \( \pi^{v/4}(c) = c \) and \( \pi^{v/4}(d) = d \) and this is a contradiction since no vertices are fixed under \( \pi^{v/4} \).

A subdesign of a \( G \)-design on \( K_v \), \( \{g_1, g_2, \ldots, g_n\} \), is a subset \( \{g'_1, g'_2, \ldots, g'_M\} \subset \{g_1, g_2, \ldots, g_n\} \) which is a \( G \)-design on some complete subgraph of \( K_v \).

Lemma 3.2 If a bicyclic LS(v) exists which admits an automorphism consisting of a cycle of length \( M \) and a cycle of length \( N \) where \( M < N \), and the design does not contain a cyclic subsystem of order \( M \) on the vertices \( \{0_0, 1_0, \ldots, (M-1)_0\} \), then \( v \equiv 9 \) (mod 24) and \( N = 2M \).

Proof. The automorphism \( \pi^M \) contains \( M \) fixed points. Suppose \( g_i \) is a block of such a design. We say an edge \((x, y)\) is absolutely fixed by \( \pi \) if \( \pi(x) = x \) and \( \pi(y) = y \). Clearly if two or three edges of \( g_i \) are absolutely fixed under an automorphism, then all edges of \( g_i \) are absolutely fixed, and therefore all vertices of \( g_i \) are fixed under the automorphism. If exactly one edge of \( g_i \) is absolutely fixed under an automorphism, then the other three edges of \( g_i \) must be interchanged. This is only possible when \( g_i = (a, b, c) - d \) where \( c \) and \( d \) are fixed and \( a \) and \( b \) are interchanged under the automorphism. Therefore, if some \( g_i \) of a bicyclic LS(v) has exactly one absolutely fixed edge under \( \pi^M \) (such a \( g_i \) exists under the hypothesis that the system does not contain a subsystem on the fixed points), it must be that \( \pi^M \) consists of \( M \) fixed points and \( N/2 \) transpositions. Therefore \( N = 2M \) and \( v \equiv 0 \) (mod 3), which implies \( v \equiv 0 \) or \( 9 \) (mod 24).

Now if \( v \equiv 0 \) (mod 24) then \( M \equiv 0 \) (mod 8) and some \( g_i \) in such a design contains the edge \((0_0, (M/2)_0)\). We see that \( g_i \) must be fixed under \( \pi^{M/2} \), a contradiction.

Lemma 3.3 If \( v \equiv 9 \) (mod 24), then there exists a bicyclic LS(v) admitting an automorphism consisting of a cycle of length \( M \) and a cycle of length \( N \) where \( N = 2M \) and \( M + N = v \).

Proof. Let \( v = 24k + 9 \), and so \( M = 8k + 3 \) and \( N = 16k + 6 \). We consider two cases based on the parity of \( k \).

\textbf{case 1.} Suppose \( k \) is odd. Consider the set of blocks:
\{(4k + 1)_1, (12k + 4)_1, 0_0\} - (4k + 1)_0
\cup \{(0_0, (3k + 3)_0, (3k + 1)_0) - (4k + 2)_0\}
\cup \{(0_0, (3k + 5)_0, (3k + 1)_0, (3k - 1)_0 - (3k + 3)_0) \text{ for } i = 0, 1, \ldots, k - 3\}
\cup \{(0_0, (5k)_0, (5k + 1)_0, (5k + 1)_0 - (6k + 2)_0) \text{ for } i = 0, 1, \ldots, k - 3\}
\cup \{(0_0, (4k - i)_1, (4k + 2 + i)_1) - (4k + 3 + 3i)_1 \text{ for } i = 0, 1, \ldots, 4k\}

**case 2.** Suppose \(k\) is even. Consider the set of blocks:

\{(4k + 1)_1, (12k + 4)_1, 0_0\} - (4k + 1)_0
\cup \{(0_0, (3k + 2)_0, (3k + 1)_0) - (4k + 1)_0\}
\cup \{(0_0, (3k + 4)_0, (3k - 2)_0 - (3k + 2)_0) \text{ for } i = 0, 1, \ldots, k - 2\}
\cup \{(0_0, (5k + 4)_0, (5k + 1)_0, (5k + 1)_0) - (6k + 1)_0 \text{ for } i = 0, 1, \ldots, k - 1\}
\cup \{(0_0, (4k - i)_1, (4k + 2 + i)_1) - (4k + 3 + 3i)_1 \text{ for } i = 0, 1, \ldots, 4k\}

In both cases, the set of blocks, along with the images of these blocks under the powers of \(\pi\), form the desired design.

Notice that under \(\pi^M\), the design given in Lemma 3.3 has \(M\) fixed points, yet there is not a subdesign on these fixed points. This is contrary to the behavior of several previously studied graph and digraph decompositions (such as Steiner triple systems [6], directed triple systems [11], and Mendelsohn triple systems [4]).

**Lemma 3.4** If a bicyclic \(LS(v)\) exists which admits an automorphism \(\pi\) consisting of a cycle of length \(M\) and a cycle of length \(N\) where \(M < N\) and when \(\pi\) is restricted to \(\{0_0, 1_0, \ldots, (M - 1)_0\}\) we have a cyclic subsystem of order \(M\) on these points, then \(M \equiv 1 \pmod{8}\) and \(N = kM\) where \(k \equiv 7 \pmod{8}\).

**Proof.** Since there is a cyclic subsystem of order \(M\), \(M \equiv 1 \pmod{8}\) is necessary by Theorem 2.1. In such a design, there must be a block of one of the following forms: \((a_1, b_1, c_1) - d_0\), \((a_1, b_1, c_0) - d_1\), or \((a_0, b_1, c_1) - d_1\).
The points of \(\{0_1, 1_1, \ldots, (N - 1)_1\}\) are fixed under \(\pi^N\) and so the images of these blocks are respectively \((a_1, b_1, c_1) - \pi^N(d_0)\), \((a_1, b_1, \pi^N(c_0)) - d_1\), and \((\pi^N(a_0), b_1, c_1) - d_1\). In each case, \(\pi^N\) must fix vertices of \(\{0_0, 1_0, \ldots, (M - 1)_0\}\) and so \(M \mid N\). If \(N\) is an even multiple of \(M\), then the edge \((0_1, (N/2)_1)\) must be in some block of the design, and again as in Theorem 2.1, we get a contradiction. Therefore \(N\) must be an odd multiple of \(M\). This condition, along with the fact that \(v = M + N \equiv 0\) or \(1 \pmod{8}\), implies that \(N = kM\) where \(k \equiv 7 \pmod{8}\).

We now show the necessary conditions of Lemmas 3.1, 3.2 and 3.4 are in fact sufficient.
Theorem 3.1 A bicyclic $L(S(v)$ admitting an automorphism consisting of
a cycle of length $M$ and a cycle of length $N$, where $M \leq N$, exists if and
only if
(i) $N = 2M$ and $v = M + N \equiv 9 \pmod{24}$, or
(ii) $M \equiv 1 \pmod{8}$ and $N = kM$ where $k \equiv 7 \pmod{8}$.

Proof. Sufficiency for (i) is given in Lemma 3.3. Therefore, we need only show sufficiency in (ii). We do so in two cases.

case 1. Suppose $M \equiv 1 \pmod{8}$ and $k \equiv 7 \pmod{16}$. Consider the set of blocks:

\[
\left\{ \left(0, \left(\frac{M-5}{4} - i\right) + \left(\frac{M(2k-1)-1}{4} + i\right) \right) \right\}_{i=0,1,\ldots,\frac{M-5}{4}}
\]

\[\cup \left\{ \left(0, \left(\frac{3M-7}{4} - i\right) + \left(\frac{M(2k+1)+1}{4} + i\right) \right) \right\}_{i=0,1,\ldots,\frac{M-5}{4}}
\]

\[\cup \left\{ \left(0, \left(\frac{3M(k-2)+9}{16} - i\right) + \left(\frac{3M(k-2)+25}{16} + i\right) \right) \right\}_{i=0,1,\ldots,\frac{M-5}{4}}
\]

\[\cup \left\{ \left(0, \left(\frac{5M(k-2)+15}{16} - i\right) + \left(\frac{5M(k-2)+47}{16} + i\right) \right) \right\}_{i=0,1,\ldots,\frac{M-5}{4}}
\]

\[\cup \left\{ \left(0, \left(\frac{3M(k-2)-7}{16} - i\right) + \left(\frac{3M(k-2)+41}{16} + i\right) \right) \right\}_{i=0,1,\ldots,\frac{M-5}{4}}
\]

\[\cup \left\{ \left(0, \left(\frac{5M(k-2)-1}{16} - i\right) + \left(\frac{5M(k-2)+63}{16} + i\right) \right) \right\}_{i=0,1,\ldots,\frac{M-5}{4}}
\]

\[\cup \left\{ \left(0, \left(\frac{M(k-2)-9}{16} - i\right) + \left(\frac{M(k-2)+25}{16} + i\right) \right) \right\}_{i=0,1,\ldots,\frac{M-5}{4}}
\]

\[\cup \left\{ \left(0, \frac{M(k-2)+1}{2} - i\right) + \left(\frac{M(2k-1)+3}{2} + i\right) \right\}_{i=0,1,\ldots,\frac{M-5}{4}}
\]

case 2. Suppose $M \equiv 1 \pmod{8}$ and $k \equiv 15 \pmod{16}$. Consider the set of blocks:

\[
\left\{ \left(0, \left(\frac{M-5}{4} - i\right) + \left(\frac{M(2k-1)-1}{4} + i\right) \right) \right\}_{i=0,1,\ldots,\frac{M-5}{4}}
\]

\[\cup \left\{ \left(0, \left(\frac{3M-7}{4} - i\right) + \left(\frac{M(2k+1)+1}{4} + i\right) \right) \right\}_{i=0,1,\ldots,\frac{M-5}{4}}
\]

\[\cup \left\{ \left(0, \left(\frac{M(k-2)-9}{16} - i\right) + \left(\frac{M(k-2)+25}{16} + i\right) \right) \right\}_{i=0,1,\ldots,\frac{M-5}{4}}
\]

\[\cup \left\{ \left(0, \frac{M(k-2)+1}{2} - i\right) + \left(\frac{M(2k-1)+3}{2} + i\right) \right\}_{i=0,1,\ldots,\frac{M-5}{4}}
\]

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for $i = 0, 1, \ldots, \frac{M-5}{4}$

\[ \bigcup \left\{ \left( 0_1, \left( \frac{3M(k-2)+17}{16} \right)_1, \left( \frac{3M(k-2)+33}{16} \right)_1 - \left( \frac{3M(k-5)+45}{16} \right)_0 \right) \right\} \]

\[ \bigcup \left\{ \left( 0_1, \left( \frac{5M(k-2)+23}{16} \right)_1, \left( \frac{5M(k-2)+55}{16} \right)_1 - \left( \frac{5M(k-2)+47}{8} \right)_1 \right) \right\} \]

\[ \bigcup \left\{ \left( 0_1, \left( \frac{3M(k-2)+1}{16} - i \right)_1, \left( \frac{3M(k-2)+49}{16} + i \right)_1 - \left( \frac{9M(k-2)+99}{16} + 2i \right)_1 \right) \right\}, \text{for } i = 0, 1, \ldots, \frac{M(k-2)-21}{16} \right\} \]

\[ \bigcup \left\{ \left( 0_1, \left( \frac{5M(k-2)+7}{16} - i \right)_1, \left( \frac{5M(k-2)+71}{16} + i \right)_1 - \left( \frac{12M(k-2)+116}{16} + 2i \right)_1 \right) \right\}, \text{for } i = 0, 1, \ldots, \frac{M(k-2)-37}{16} \right\}. \]

In both cases, the set of blocks, along with the images of these blocks under the powers of $\pi$ and a set of blocks for a cyclic $LS(M)$ on the point set $\{0_0, 1_0, \ldots, (M-1)_0\}$, form the desired design.

References


