Bicyclic Decompositions of the Complete Digraph into each of the Orientations of a 4-Cycle

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Abstract. A decomposition of a digraph is said to be bicyclic if it admits an automorphism consisting of exactly two disjoint cycles. Necessary and sufficient conditions are given for the existence of bicyclic decompositions of the complete digraph into each of the four orientations of a 4-cycle.

1 Introduction

Let D_v denote the complete digraph on v vertices. If g is a digraph, then a *g-decomposition* of D_v is a set $\gamma = \{g_1, g_2, \ldots, g_n\}$ if arc-disjoint subgraphs of D_v, called *blocks*, each of which is isomorphic to g and such that $\bigcup_{i=1}^{n} A(g_i) = A(D_v)$, where $A(G)$ is the arc set of digraph G. An *automorphism* of a g-decomposition of D_v is a permutation of the vertex set of D_v which fixes the set γ. The *orbit* of a block g under the automorphism π is the set $\{\pi^n(g) \mid n \in \mathbf{Z}\}$ and the *length* of an orbit is its cardinality.

There are two orientations of the 3-cycle: the 3-circuit and the digraph (called a "transitive triple"):

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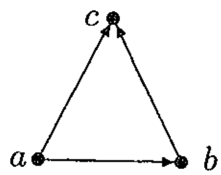
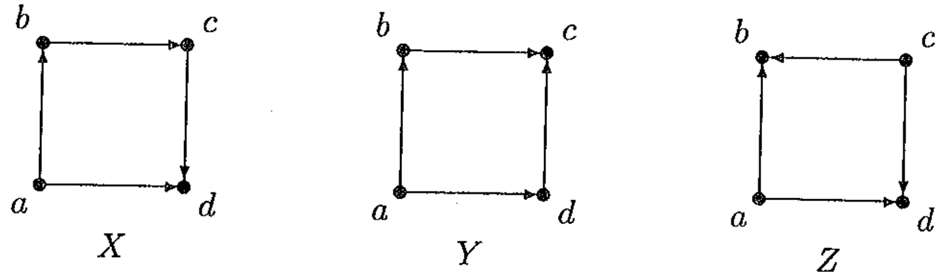


A decomposition of D_v into 3-circuits is equivalent to a Mendelsohn triple system of order v, denoted $MTS(v)$ [10]. A decomposition of D_v into transitive triples is equivalent to a directed triple system of order v, denoted $DTS(v)$ [9].

There are four orientations of the 4-cycle: the 4 circuit and the following:



We represent X as $[a, b, c, d]_X$, Y as $[a, b, c, d]_Y$, and Z as $[a, b, c, d]_Z$. We represent the 4-circuit with arc set $\{(a, b), (b, c), (c, d), (d, a)\}$ by any cyclic shift of $[a, b, c, d]_C$. A 4-circuit decomposition of D_v exists if and only if $v \equiv 0$ or 1 (mod 4), $v \neq 4$ [13]. A X-decomposition of D_v exists if and only if $v \equiv 0$ or 1 (mod 4), $v \neq 5$, a Y-decomposition of D_v exists if and only if $v \equiv 0$ or 1 (mod 4), $v \notin \{4, 5\}$, and a Z-decomposition of D_v exists if and only if $v \equiv 1$ (mod 4) [8].

A digraph decomposition admitting an automorphism consisting of a single cycle is said to be *cyclic*. A cyclic $MTS(v)$ exists if and only if $v \equiv 1$ or 3 (mod 6), $v \neq 9$ [4], and a cyclic $DTS(v)$ exists if and only if $v \equiv 1$, 4, or 7 (mod 12) [5]. A cyclic 4-circuit decomposition of D_v exists if and only if $v \equiv 1$ (mod 4) [12], a cyclic X-decomposition of D_v exists if and only if $v \equiv 1$ (mod 4), $v \neq 5$, a cyclic Y-decomposition of D_v exists if and only if $v \equiv 1$ (mod 4), $v \neq 5$, and a cyclic Z-decomposition of D_v exists if and only if $v \equiv 1$ (mod 4) [3,11]. A digraph decomposition of D_v admitting an automorphism consisting of two disjoint cycles is said to be *bicyclic*. Bicyclic Steiner triple systems (a Steiner triple system of order v is equivalent to a D_3-decomposition of D_v) are explored in [1,2,6]. Necessary and sufficient conditions for the existence of a bicyclic $DTS(v)$ are given in [7]. The purpose of this paper is to give necessary and sufficient conditions for the existence of a bicyclic g-decomposition of D_v where g is an orientation of the 4-cycle.

2 Automorphism Consists of Two Cycles of Equal Length

In this section, we give necessary and sufficient conditions for the existence of a g-decomposition of D_v, where g is an orientation of the 4-cycle, admitting an automorphism consisting of two disjoint cycles of equal length. Clearly, v must be even in such a decomposition. Throughout this section we suppose the vertex set of D_v is $\{0_0, 1_0, \ldots, (v/2-1)_0, 0_1, 1_1, \ldots, (v/2-1)_1\}$ and let the relevant automorphism be $(0_0, 1_0, \ldots, (v/2-1)_0)(0_1, 1_1, \ldots, (v/2-1)_1)$.

Theorem 2.1 *A bicyclic 4-circuit decomposition of D_v admitting an automorphism consisting two cycles each of length $v/2$ exists if and only if $v \equiv 0$ (mod 8).*

Proof. Since v must be even, $v = 0$ (mod 4) is clearly necessary. We show that such a system does not exist for $v \equiv 4$ (mod 8). Suppose such a system does exist. We associate a *difference* with each arc of D_v as follows:

arc: (a_0, b_0),	difference: $b - a$ (mod $v/2$) of type 0
arc: (a_1, b_1),	difference: $b - a$ (mod $v/2$) of type 1
arc: (a_0, b_1),	difference: $b - a$ (mod $v/2$) of type 01
arc: (a_1, b_0),	difference: $b - a$ (mod $v/2$) of type 10.

The differences of types 0 and 1 are sometimes called *pure differences* and the differences of types 01 and 10 are sometimes called *mixed differences.* Notice that in our hypothesized design, the length of the orbit of a block must be either $v/2$ (in which case there are four distinct differences associated with the arcs of the block and the sum of these differences is 0 modulo $v/2$) or $v/4$ (in which case there are two distinct differences associated with the arcs of the block and the sum of these differences is $v/4$ modulo $v/2$). Since the number of blocks in such a design is $v(v-1)/4 \equiv v/4$ (mod $v/2$), there must be an odd number of orbits of length $v/4$. The sets of differences are

pure type 0:	$\{1, 2, \ldots, v/2-1\}$
pure type 1:	$\{1, 2, \ldots, v/2-1\}$
mixed type 01:	$\{0, 1, 2, \ldots, v/2-1\}$
mixed type 10:	$\{0, 1, 2, \ldots, v/2-1\}$.

The total sum of these differences is $v(v-2)/2 \equiv 0$ (mod $v/2$). Therefore there must be an even number of orbits of length $v/4$. This is a contradiction and the hypothesized design does not exist.

We now show that the condition $v \equiv 0 \pmod 8$ is sufficient for the desired design to exist. First, for the case $v = 8$, consider the blocks:

$$\{[0_0, 0_1, 1_0, 2_1]_C, [0_0, 3_1, 1_1, 0_1]_C, [0_1, 3_0, 1_0, 2_0]_C,$$

$$[0_1, 1_1, 2_1, 3_1]_C, [0_0, 3_0, 2_0, 1_0]_C\}.$$

These blocks, along with their images under the powers of π, form the desired design. Now for $v > 8$, consider the set of blocks:

$$A = \{[0_0, (v/8)_0, (v/4)_0, (3v/8)_0]_C, [0_0, (3v/8)_0, (v/4)_0, (v/8)_0]_C\}$$
$$\bigcup\{[0_0, (v/4)_0, (v/4+2)_1, 1_1]_C, [0_0, (v/2-2)_1, (v/4-2)_1, (v/2-1)_1]_C\}$$
$$\bigcup\{[0_0, 0_1, (v/4)_0, (v/4)_1]_C, [0_0, (v/4)_1, (v/4)_0, 0_1]_C\}$$
$$\bigcup\{[0_0, 1_1, 3_0, 2_1]_C, [0_0, (v/4-1)_1, (v/2-2)_0, (v/4-1)_1]_C\}$$
$$\bigcup\{[0_1, (1+2s)_1, (3+4s)_1, (2+2s)_1]_C \mid s \in \mathbf{Z}_{(v-8)/8}\}$$
$$\bigcup\{[0_0, (3+2s)_1, (7+4s)_0, (4+2s)_1]_C, [0_1, (3+2s)_0, (7+4s)_1, (4+2s)_0]_C \mid s \in \mathbf{Z}_{(v-16)/8}\}$$

We establish sufficiency in two cases:

Case 1. Suppose $v \equiv 0 \pmod{16}$. Consider the set of blocks:

$$A \bigcup\{[0_0, (v/8-1)_0, (v/4)_0, (v/8+1)_0]_C\}$$
$$\bigcup\{[0_0, (1+2s)_0, (3+4s)_0, (2+2s)_0]_C \mid s \in \mathbf{Z}_{(v-16)/16}\}$$
$$\bigcup\{[0_0, (v/8+2+2s)_0, (v/4+5+4s)_0, (v/8+3+2s)_0]_C \mid s \in \mathbf{Z}_{(v-16)/16}\}.$$

Case 2. Suppose $v \equiv 8 \pmod{16}$, $v > 8$. Consider the set of blocks:

$$A \bigcup\{[0_0, (1+2s)_0, (3+4s)_0, (2+2s)_0]_C \mid s \in \mathbf{Z}_{(v-8)/16}\}$$
$$\bigcup\{[0_0, (v/8+1+2s)_0, (v/4+3+4s)_0, (v/8+2+2s)_0]_C \mid s \in \mathbf{Z}_{(v-8)/16}\}.$$

In both cases, the given blocks, along with their images under the powers of π, form the desired design. ∎

Theorem 2.2 *Neither a bicyclic X−decomposition of D_v nor a bicyclic Y−decomposition of D_v exists which admit an automorphism consisting of two cycles each of length $v/2$.*

Proof. Suppose such designs exist. The length of the orbit of each block in such a system is $v/2$. Therefore $v/2$ must divide the total number of blocks $v(v-1)/4$. This implies that v is odd, an obvious contradiction. ∎

Since a Z-decomposition of D_v only exists for $v \equiv 1 \pmod 4$, we clearly have:

Theorem 2.3 *A bicyclic Z-decomposition of D_v admitting an automorphism consisting of two cycles of length $v/2$ does not exist.*

3 Automorphism Consists of Two Cycles of Different Lengths

In this section we consider g-decompositions of D_v, where g is an orientation of the 4-cycle, admitting an automorphism consisting of two disjoint cycles, one of length M and one of length N, where $M < N$. Throughout this section, we suppose the vertex set of D_v is $\{0_0, 1_0, \ldots, (M-1)_0, 0_1, 1_1, \ldots, (N-1)_1\}$ and let the relevant automorphism be $(0_0, 1_0, \ldots, (M-1)_0)(0_1, 1_1, \ldots, (N-1)_1)$. First, we prove some necessary conditions.

Lemma 3.1 *The fixed points of an automorphism π of a g-decomposition of D_v, where g is an orientation of the 4-cycle, form a subsystem of the g-decomposition. That is, if $\pi(a) = a$, $\pi(b) = b$ and (a, b) is an arc of some g_{ab} of the g-decomposition, then $\pi(c) = c$ and $\pi(d) = d$ where the vertex set of g_{ab} is $\{a, b, c, d\}$.*

Proof. Let g_{ab} be an element of a g-decomposition of D_v where g is an orientation of the 4-cycle and suppose $\pi(a) = a$, $\pi(b) = b$, and (a, b) is an arc of g_{ab}. Since arc (a, b) is in only one element of the g-decomposition, and $\pi((a, b)) = (a, b)$, it must be that $\pi(g_{ab}) = g_{ab}$. This can only happen when the other two vertices of g_{ab} are fixed. ∎

Lemma 3.2 *If a bicyclic g-decomposition of D_v exists, where g is an orientation of the 4-cycle, admitting an automorphism consisting of two disjoint cycles of lengths M and N, where $M < N$, then $M \equiv 1 \pmod 4$ and $M \mid N$.*

Proof. Let π be an automorphism of such a g-decomposition where π consists of two cycles as described. Then π^M has M fixed points and by Lemma 3.1 we see that there must be a cyclic subsystem of the given decomposition. Therefore $M \equiv 1 \pmod 4$.

Suppose each arc of the form (a_1, b_1) is in some copy of g whose vertex set is of the form $\{a_1, b_1, c_1, d_1\}$. Then the system has a cyclic subsystem of order N and $N \equiv 1 \pmod 4$. However, $M \equiv N \equiv 1 \pmod 4$ and $v = M + N \equiv 2 \pmod 4$, a contradiction. Therefore, there must be some g with vertex set of the form $\{a_0, b_1, c_1, d_1\}$ and containing an arc of the form (c_1, d_1). Applying π^N to this g, we see that the vertex set of $\pi^N(g)$ is $\{\pi^N(a_0), b_1, c_1, d_1\}$ and $\pi((c_1, d_1)) = (c_1, d_1)$ is an arc of $\pi^N(g)$. As in Lemma 3.1, $\pi^N(a_0) = a_0$ and therefore $M \mid N$. ∎

We now establish necessary and sufficient conditions for the existence of a bicyclic g-decomposition of D_v.

Theorem 3.1 *A bicyclic 4-circuit decomposition of D_v admitting an automorphism consisting of a cycle of length M and a cycle of length N, where $M < N$, exists if and only if $M \equiv 1$ (mod 4), $v = M + N \equiv 0$ or 1 (mod 4), $v \neq 4$, and $M \mid N$.*

Proof. By Lemma 3.2, $M \equiv 1 \pmod 4$ and $M \mid N$. Since a 4-circuit system of order v only exists for $v \equiv 0$ or $1 \pmod 4$, $v \neq 4$, it follows that $v = M + N \equiv 0$ or $1 \pmod 4$, $v \neq 4$. For sufficiency, we consider three cases:

Case 1. Suppose $M \equiv 1 \pmod 4$, $M \mid N$ and $N \equiv 0 \pmod 8$. Consider the blocks:

$$\{0_1, (2s+1)_1, (4s+3)_1, (2s+2)_1]_C \mid s \in \mathbf{Z}_{(N-4)/4} \setminus \{N/8 - 1\}\}$$
$$\bigcup\{[0_0, 0_1, (3N/4)_1, (N/4)_1]_C, [0_1, (N/4)_1, (N/2)_1, (3N/4)_1]_C\}$$
$$\bigcup\{[0_1, (N/2-1)_1, (N/4)_1, (3N/4+1)_1]_C\}$$
$$\bigcup\{[0_1, (2s+1)_0, (4s+3)_1, (2s+2)_0]_C, [0_0, (2s+1)_1, (4s+3)_0, (2s+2)_1]_C \mid s \in \mathbf{Z}_{(M-1)/4}\}.$$

Case 2. Suppose $M \equiv 1 \pmod 4$, $M \mid N$ and $N \equiv 4 \pmod 8$. Consider the blocks:

$$\{0_1, (2s+1)_1, (4s+3)_1, (2s+2)_1]_C \mid s \in \mathbf{Z}_{(N-4)/4} \setminus \{(N-4)/8\}\}$$
$$\bigcup\{[0_0, 0_1, (3N/4)_1, (N/4)_1]_C, [0_1, (N/4)_1, (N/2)_1, (3N/4)_1]_C\}$$
$$\bigcup\{[0_1, (N/2-1)_1, (3N/4)_1, (N/4+1)_1]_C\}$$
$$\bigcup\{[0_1, (2s+1)_0, (4s+3)_1, (2s+2)_0]_C, [0_0, (2s+1)_1, (4s+3)_0, (2s+2)_1]_C \mid s \in \mathbf{Z}_{(M-1)/4}\}.$$

Case 3. Suppose $M \equiv 1 \pmod 4$, $M \mid N$ and $N \equiv 3 \pmod 4$. Consider the blocks:

$$\{[0_1, (2s+1)_1, (4s+3)_1, (2s+2)_1]_C \mid s \in \mathbf{Z}_{(N-7)/4}\}$$
$$\bigcup\{[0_0, (M-1)_1, ((N-3)/2+M)_1, M_1]_C, [0_0, 0_1, ((N-3)/2)_1, 1_1]_C\}$$
$$\bigcup\{[0_0, 1_1, (N-3)/2)_1, (N-1)_1]_C, [0_0, ((M-1)/2)_1, (M-1)_0, ((3M-1)/2)_1]_C\}$$
$$\bigcup\{[0_0, (2s+2)_1, (4s+5)_0, (2s+3)_1]_C, [0_1, (2s+2)_0, (4s+5)_1, (2s+3)_0]_C \mid s \in \mathbf{Z}_{(M-5)/4}\}.$$

In each case, the given blocks, along with their images under the powers of the automorphism π and a collection of blocks for a cyclic 4-circuit decomposition of D_M, form the desired design. ∎

Theorem 3.2 *A bicyclic X-decomposition of D_v admitting an automorphism consisting of a cycle of length M and a cycle of length N, where $M < N$, exists if and only if $M \equiv 1 \pmod 4$, $M \neq 5$, $M \mid N$ and $v = M + N \equiv 0 \pmod 4$, $v \neq 5$.*

Proof. As in Theorem 3.1, it is necessary that $M \equiv 1 \pmod 4$ and $M \mid N$. The existence condition on v for a X-decomposition of D_v also implies that $M \neq 5$ and $v = M + N \equiv 0$ or $1 \pmod 4$, $v \neq 5$. Suppose such a system exists with $v \equiv 1 \pmod 4$. This system contains a cyclic subsystem of order $M \equiv 1 \pmod 4$, and so $N \equiv 0 \pmod 4$. Therefore there are $M(M-1)/4$ blocks in this system of the form $[a_0, b_0, c_0, d_0]_X$. Blocks of any other form in this system have orbits of length N and there are

$$\frac{(M+N)(M+N-1)}{4} - \frac{M(M-1)}{4} = \frac{N(2M+N-1)}{4}$$

such blocks. Consequently, $(2M+N-1)/4$ is an integer, a contradiction. Therefore, no such system exists.

We show sufficiency in the remaining cases as follows. Define the following sets:

$$A = \{[0_0, ((M-1)/2)_1, ((M-1)/2))_0, 0_1]_X, [0_1, ((M-1)/2)_0, ((M-1)/8)_1, ((5M+3)/8)_0]_X\}$$
$$\bigcup\{[0_0, ((M-3)/2-s)_1, ((M-1)/2)_0, (s+1)_1]_X, [0_1, ((M-3)/2 - s)_0, ((M-5)/2-2s)_1, (M-1-s)_0]_X \mid s \in \mathbf{Z}_{(M-9)/8}\}$$
$$\bigcup\{[0_0, ((3M-11)/8-s)_1, ((M-3)/2)_0, ((M-1)/8+s)_1]_X, [0_1, ((3M-11)/8-s)_0, ((M-5)/4-2s)_1, ((7M+1)/8-s)_0]_X \mid s \in \mathbf{Z}_{(M-1)/8}\},$$

$$B = \{[0_0, ((M-1)/2-s)_1, ((M-1)/2+s)_0, (2s)_1]_X, [0_1, ((M-1)/2 -s)_0, ((M+1)/2+s)_1, (1+2s)_0]_X \mid s \in \mathbf{Z}_{(M+3)/8}\}$$

$$\bigcup\{[0_0, ((3M-7)/8-s)_1, ((M-5)/8)_0, ((3M+1)/4+s)_1]_X, [0_1, ((3M-7)/8-s)_0, ((M-1)/4)_1, ((7M+5)/8+s)_0]_X \mid s \in \mathbf{Z}_{(M-5)/8}\}.$$

$$C = \{[0_1, ((N-3)/2-s)_1, (N-1)_1, ((N-11)/4-2s)_1]_X \mid s \in \mathbf{Z}_{(N-11)/8}\}$$

$$\bigcup\{[0_1, ((3N-1)/8-s)_1, (N-1)_1, (N-3-2s)_1]_X \mid s \in \mathbf{Z}_{(N-3)/8}\}$$

$$\bigcup\{[0_1, 1_1, ((N+1)/4)_1, (N-1)_1]_X\},$$

$$D = \{[0_1, ((N-3)/2-s)_1, (N-1)_1, ((N-11)/4-2s)_1]_X, [0_1, ((3N-5)/8-s)_1, (N-1)_1, (N-4-2s)_1]_X \mid s \in \mathbf{Z}_{(N-7)/8}\}$$

$$\bigcup\{[0_1, ((N-3)/4)_1, ((N-1)/2)_1, (N-1)_1]_X\},$$

We consider 4 cases:

Case 1. Suppose $M \equiv 1 \pmod 8$ and $N \equiv 3 \pmod 8$. Consider the set of blocks: $A \bigcup C \bigcup \{[0_1, ((3M-3)/8)_0, ((N-3)/4)_1, ((N-1)/2)_1]_X\}$.

Case 2. Suppose $M \equiv 1 \pmod 8$ and $N \equiv 7 \pmod 8$. Consider the set of blocks: $A \bigcup D \bigcup \{[0_1, ((3M-3)/8)_0, ((N-3)/4)_1, (N-2)_1]_X\}$.

Case 3. Suppose $M \equiv 5 \pmod 8$ and $N \equiv 3 \pmod 8$. Consider the set of blocks: $B \bigcup C \bigcup \{[0_1, ((7M-3)/8)_0, ((N-3)/4)_1, ((N-1)/2)_1]_X\}$.

Case 4. Suppose $M \equiv 5 \pmod 8$ and $N \equiv 7 \pmod 8$. Consider the set of blocks: $B \bigcup D \bigcup \{[0_1, ((7M-3)/8)_0, ((N-3)/4)_1, (N-2)_1]_X\}$.

In each case, the given blocks, along with their images under the powers of the automorphism π and a collection of blocks for a cyclic X-decomposition of D_M, form the desired design. ∎

Theorem 3.3 *A bicyclic Y-decomposition of D_v admitting an automorphism consisting of a cycle of length M and a cycle of length N, where $M < N$, exists if and only if $M \equiv 1 \pmod 4$, $M \geq 9$, $M \mid N$ and $v = M + N \equiv 0 \pmod 4$, $v \neq 4$.*

Proof. The necessary conditions follow as in Theorem 3.2. We now establish sufficiency. Consider the blocks:

$\{[0_1, (4s+3)_1, (8s+9)_1, (4s+5)_1]_Y \mid s \in \mathbf{Z}_{(N-3)/4}\}$

$\{[0_0, ((M+5)/2)_1, (M+3)_0, ((M+1)/2)_1]_Y, [0_0, ((M+7)/2)_1, (M+3)_0, ((M-1)/2)_1]_Y, [0_1, ((M+3)/2)_0, 3_1, 2_1]_Y\}$

$\bigcup\{[0_0, (1+2s)_1, (1+4s)_0, (2s)_1]_Y \mid s \in \mathbf{Z}_{(M-1)/4}\}$

$\bigcup\{[0_0, (M-2-2s)_1, (M-3-4s)_0, (M-1-2s)_1]_Y \mid s \in \mathbf{Z}_{(M-9)/4}\}.$

The given blocks, along with their images under the powers of the automorphism π and a collection of blocks for a cyclic Y-decomposition of D_m, form the desired design. ∎

Theorem 3.4 *A bicyclic Z-decomposition of D_v admitting an automorphism consisting of a cycle of length M and a cycle of length N, where $M < N$ does not exist.*

Proof. As in Theorem 3.2, $N \equiv 0 \pmod 4$ is not possible and $N \equiv 3 \pmod 4$ is necessary. However, this implies that $v = M + N \equiv 0 \pmod 4$ and such a system does not exist. ∎

Necessary and sufficient conditions for the existence of a bicyclic g-decomposition of D_v, where g is an orientation of the 4-cycle, are given in Theorems 2.1–2.3 and 3.1–3.4.

4 References

1. R. Calahan and R. Gardner, A special case of bicyclic Steiner triple systems, *Congressus Numerantium* **88** (1992), 77–80.

2. R. Calahan-Zijlstra and R. Gardner, Bicyclic Steiner triple systems, *Discrete Math.* **128** (1994), 35–44.

3. G. D. Coker, R. Gardner and R. Johnson, Decompositions of the complete digraph and the complete graph which admit certain automorphisms, *Congressus Numerantium* **108** (1995), 65–84.

4. C. Colbourn and M. Colbourn, Disjoint cyclic Mendelsohn triple systems, *Ars.Combinatoria* **11** (1981), 3–8.

5. M. Colbourn and C. Colbourn, The analysis of directed triple systems by refinement, *Annals of Discrete Math.* **15** (1982), 97–103.

6. R. Gardner, A note on a class of Steiner triple systems, *Ars Combinatoria* **36** (1993), 157–160.

7. R. Gardner, Bicyclic directed triple systems, *Ars Combinatoria* **49** (1998), 249–257.

8. F. Harary, W. Wallis, and K. Heinrich, Decompositions of complete symmetric digraphs into the four oriented quadrilaterals, *Combinatorial Mathematics* in Lecture Notes in Mathematics #686, Springer-Verlag, 1978.

9. S. Hung and N. Mendeslohn, Directed triple systems, *J. Combin. Theory Series A* **14** (1973), 310–318.

10. N. Mendelsohn, A natural generalization of Steiner triple systems, *Computers in Number Theory*, eds. A. O. Atkin and B. Birch, Academic Press, London, 1971.

11. B. Micale and M. Pennisi, Cyclic decompositions of complete symmetric digraphs into the four oriented quadrilaterals, *Utilitas Mathematica*, to appear.

12. B. Micale and M. Pennisi, Cyclic Mendeslohn quadruple systems, *Ars Combinatoria* **35** (1993), 225–236.

13. J. Schönheim, Partition of the edges of the directed complete graph into 4-cycles, *Discrete Math.* **11** (1975), 67–70.