Cyclic and Rotational Decompositions of $K_n$ into Stars

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Abstract. We give necessary and sufficient conditions for the existence of a decomposition of the complete graph into stars which admits either a cyclic or a rotational automorphism.

1 Introduction

We denote the complete graph on $n$ vertices by $K_n$ and the star with $m$ edges by $S_m$. Let $m_1 \geq m_2 \geq \ldots \geq m_l$ be nonnegative integers. Then a $S_{m_1}, S_{m_2}, \ldots, S_{m_l}$ decomposition of $K_n$ (or a star decomposition of $K_n$, for short) is a collection of stars such that

$$E(S_{m_i}) \cap E(S_{m_j}) = \emptyset \text{ if } i \neq j, \text{ and } \bigcup_{i=1}^{l} E(S_{m_i}) = E(K_n).$$

It was recently shown in [2] that such a decomposition exists if and only if

$$\sum_{i=1}^{k} m_i \leq \sum_{i=1}^{k} (n - i) \text{ for } k = 1, 2, \ldots, n - 1, \text{ and } \sum_{i=1}^{l} m_i = \binom{n}{2}.$$

An automorphism of a star decomposition is a permutation of $V(K_n)$ which fixes the set $\{S_{m_1}, S_{m_2}, \ldots, S_{m_l}\}$. The orbit of a star under an automorphism $\pi$ is the collection of images of the star under the powers of $\pi$. A permutation of $V(K_n)$ which consists of a single cycle of length $n$ is said to be cyclic. A permutation of $V(K_n)$ consisting of a fixed point and a cycle of length $n - 1$ is said to be rotational. Several graph and digraph decompositions have been studied which admit either a cyclic or rotational automorphism. See, for example, [1, 3, 4, 5]. The purpose of this paper is to give necessary and sufficient conditions for the existence of star decompositions of $K_n$ which admit either a cyclic automorphism or a rotational automorphism.

2 Cyclic Star Decompositions of $K_n$

Throughout this section, we assume the vertex set of $K_n$ is $\{0, 1, \ldots, n-1\}$ and we will construct star decompositions of $K_n$ admitting $\pi = (0, 1, \ldots, n-1)$
Lemma 2.1 If there exists a $S_{m_1}, S_{m_2}, \ldots, S_{m_k}$-decomposition of $K_n$ which admits a cyclic automorphism and if $n$ is even, then $|\{i \mid m_i = 1\}| \equiv n/2 \pmod{n}$.

Proof. The edge $(0,n/2)$ must lie in some star, say $S_{m_s}$. Then $\pi^{n/2}((0,n/2)) = (0,n/2)$ and since each edge occurs in exactly one star of the decomposition, it must be that $\pi^{n/2}(S_{m_s}) = S_{m_s}$. Therefore $m_s = 1$. Let $A = \{\pi^i(S_{m_i}) \mid i \in \mathbb{Z}\}$. Then $|A| = n/2$ and if $S_{m_t} \notin A$ then the length of the orbit of $S_{m_t}$ is $n$. Therefore $|\{i \mid m_i = 1\}| \equiv n/2 \pmod{n}$.

As argued in Lemma 2.1, the length of the orbit of every star in a cyclic star decomposition of $K_n$ is $n$ except for the special "short orbit" stars in set $A$. We therefore have:

Lemma 2.2 If there exists a $S_{m_1}, S_{m_2}, \ldots, S_{m_k}$-decomposition of $K_n$ which admits a cyclic automorphism, then for $k = 1, 2, \ldots, n-1$, $|\{i \mid m_i = k\}| \equiv 0 \pmod{n}$, except for the case $k = 1$ when $n$ is even.

We show the necessary conditions of Lemmas 2.1 and 2.2, along with the necessary conditions for the existence of a star decomposition of $K_n$, are sufficient for the existence of a cyclic star decomposition of $K_n$.

Theorem 2.1 Let $m_1 \geq m_2 \geq \cdots \geq m_k$ be nonnegative integers. Then there is a cyclic $S_{m_1}, S_{m_2}, \ldots, S_{m_k}$-decomposition of $K_n$ if and only if

$$\sum_{i=1}^{k} m_i \leq \sum_{i=1}^{k} (n-i) \text{ for } k = 1, 2, \ldots, n-1, \quad \sum_{i=1}^{l} m_i = \binom{n}{2}$$

and

(a) $|\{i \mid m_i = k\}| \equiv 0 \pmod{n}$ for all $k = 1, 2, \ldots, n-1$ if $n$ is odd, or
(b) $|\{i \mid m_1 = 1\}| \equiv n/2 \pmod{n}$ and $|\{i \mid m_i = k\}| \equiv 0 \pmod{n}$ for all $k = 2, 3, \ldots, n-1$ if $n$ is even.

Proof. We need only establish sufficiency. Without loss of generality, we may assume $m_i \geq 1$. If $n$ is odd, consider the collection of stars with edge sets

$$E(S_{m_1-kn-1}) = \{(i, i + r + \sum_{j=1}^{k} m_{l-(j-1)n}) \mid r = 1, 2, \ldots, m_{l-1} \}$$

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for \( i = 0, 1, \ldots, n - 1 \) and \( k = 0, 1, \ldots, \lfloor l/n \rfloor - 1 \). If \( n \) is even, consider the collection of stars with edge sets

\[
E(S_{m_i}) = \{(i, i + n/2)\}
\]

for \( i = 0, 1, \ldots, n/2 - 1 \), and

\[
E(S_{m_{i-n/2-1}}) = \{(i, i+r+\sum_{j=1}^{k} m_{i-n/2-(j-1)n}) \mid r = 1, 2, \ldots, m_{i-n/2-1-kn}\}
\]

for \( i = 0, 1, \ldots, n - 1 \) and \( k = 0, 1, \ldots, (l - n/2)/n - 1 \). In each case, the given collection of stars forms a cyclic star decomposition of \( K_n \).

### 3 Rotational Star Decompositions of \( K_n \)

Throughout this section, we assume the vertex set of \( K_n \) is \( \{\infty, 0, 1, \ldots, n-2\} \) and we will construct star decompositions of \( K_n \) admitting \( \pi = (\infty) \) \((0,1,\ldots,n-2)\) as an automorphism.

As in Lemma 2.1, if \( n - 1 \) is even, then the edge \((0, (n - 1)/2)\) must occur in some \( S_{m_s} \) where \( m_s = 1 \). We analogously have:

**Lemma 3.1** If there exists a \( S_{m_1}, S_{m_2}, \ldots, S_{m_i} \)-decomposition of \( K_n \) which admits a rotational automorphism and if \( n \) is odd, then \( |\{i \mid m_i = 1\}| \equiv (n - 1)/2 \pmod{n - 1} \).

The orbit of each star of a rotational star decomposition of \( K_n \) is of length \( n - 1 \), with two possible types of exceptions: (1) if \( n \) is odd, then the stars \( S_1 \) with edge sets \( \{(i, i + (n - 1)/2)\} \) for some \( i \) have orbits of length \( (n - 1)/2 \), and (2) if \( m \mid (n - 1) \), \( m \neq 1 \), say \( (n - 1)/m = p \) then the stars \( S_m \) with edge sets \( \{(\infty, i), (\infty, i+p), \ldots, (\infty, i+n-1-p)\} \) for some \( i \) have orbits of length \( p \).

**Theorem 3.2** Let \( m_1 \geq m_2 \geq \cdots \geq m_l \) be nonnegative integers. Then there is a rotational \( S_{m_1}, S_{m_2}, \ldots, S_{m_l} \)-decomposition of \( K_n \) if and only if

\[
\sum_{i=1}^{k} m_i \leq \sum_{i=1}^{k} (n-i) \text{ for } k = 1, 2, \ldots, n-1, \quad \sum_{i=1}^{l} m_i = \binom{n}{2}
\]

and

(a) \( |\{i \mid m_i = k\}| \equiv 0 \pmod{n - 1} \) for all \( k = 1, 2, \ldots, n - 1 \) if \( n \) is even,
(b) \(|\{i \mid m_i = 1\}| \equiv (n-1)/2 \ (\text{mod} \ n-1)\) and \(|\{i \mid m_i = k\}| \equiv 0 \ (\text{mod} \ n-1)\) for all \(k = 2, 3, \ldots, n-1\) if \(n\) is odd, or

(c) if \(m \mid (n-1)\), say \((n-1)/m = p\), for some \(m \in \{m_1, m_2, \ldots, m_l\}\), \(m \neq 1\), then \(|\{i \mid m_i = m\}| \equiv p \ (\text{mod} \ n-1)\) and \(|\{i \mid m_i = k\}| \equiv 0 \ (\text{mod} \ n-1)\) for all \(k = 1, 2, \ldots, m-1, m+1, \ldots, n-1\) if \(n\) is even, or

(d) if \(m \mid (n-1)\), say \((n-1)/m = p\), for some \(m \in \{m_1, m_2, \ldots, m_l\}\), \(m \neq 1\), then \(|\{i \mid m_i = m\}| \equiv p \ (\text{mod} \ n-1)\), \(|\{i \mid m_i = 1\}| \equiv (n-1)/2 \ (\text{mod} \ n-1)\) and \(|\{i \mid m_i = k\}| \equiv 0 \ (\text{mod} \ n-1)\) for all \(k = 2, 3, \ldots, m-1, m+1, \ldots, n-1\) if \(n\) is odd.

**Proof.** We need only establish sufficiency. Without Loss of generality, we may assume \(m_i \geq 1\). We consider the four cases separately.

(a) Consider the collection of stars with edge sets

\[ E(S_{m_{i-1}}) = \{(\infty, i)\} \bigcup \{(i, i+r) \mid r = 1, 2, \ldots, m_i - 1\} \]

for \(i = 0, 1, \ldots, n-2\) and

\[ E(S_{m_{i-k(n-1)-1}}) = \{(i, i+r - 1 + \sum_{j=1}^{k} m_{i-(j-1)(n-1)}) \mid r = 1, 2, \ldots, m_i - k(n-1)\} \]

for \(i = 0, 1, \ldots, n-2\) and \(k = 1, 2, \ldots, l/(n-1) - 1\).

(b) Consider the collection of stars with edge sets

\[ E(S_{m_{i-1}}) = \{(i, i + (n-1)/2)\} \]

for \(i = 0, 1, \ldots, (n-1)/2 - 1\),

\[ E(S_{m_{i-(n-1)/2-i}}) = \{(\infty, i)\} \bigcup \{(i, i+r) \mid r = 1, 2, \ldots, m_{i-(n-1)/2-k(n-1)-1}\} \]

for \(i = 0, 1, \ldots, n-2\), and

\[ E(S_{m_{i-(n-1)/2-k(n-1)-i}}) = \{(i, i+r - 1 + \sum_{j=1}^{k} m_{i-(n-1)/2-(j-1)(n-1)}) \mid r = 1, 2, \ldots, m_{l-(n-1)/2-k(n-1)}\} \]

for \(i = 0, 1, \ldots, n-2\) and \(k = 1, 2, \ldots, (l - (n-1)/2)/(n-1) - 1\).
(c) Let $t$ be the largest index such that $m_t = m$. Consider the collection of stars with edge sets

$$E(S_{m_t-k(n-1)-i}) = \{(i, i + r + \sum_{j=1}^{k} m_{t-(j-1)(n-1)}) \mid r = 1, 2, \ldots, m_{t-k(n-1)} \}$$

for $i = 0, 1, \ldots, n - 2$ and $k = 0, 1, \ldots, (l - t)/(n - 1) - 1$,

$$E(S_{m_{t-i}}) = \{(\infty, i + rp) \mid r = 0, 1, \ldots, m_t - 1 \}$$

for $i = 0, 1, \ldots, p - 1$,

$$E(S_{m_{t-p-k(n-1)-i}}) = \{(i, i + r + \sum_{j=1}^{(l-t)/(n-1)} m_{t-(j-1)(n-1)} + \sum_{j=1}^{k} m_{t-p-(j-1)(n-1)}) \mid r = 1, 2, \ldots, m_{t-p-k(n-1)} \}$$

for $i = 0, 1, \ldots, n - 2$ and $k = 0, 1, \ldots, (t - p)/(n - 1) - 1$.

(d) Let $t$ be the largest index such that $m_t = m$. Consider the collection of stars with edge sets

$$E(S_{m_{t-i}}) = \{(i, i + (n - 1)/2) \}$$

for $i = 0, 1, \ldots, (n - 1)/2 - 1$,

$$E(S_{m_{t-(n-1)/2-k(n-1)-i}}) = \{(i, i + r + \sum_{j=1}^{k} m_{t-(n-1)/2-(j-1)(n-1)}) \mid r = 1, 2, \ldots, m_{t-(n-1)/2-k(n-1)} \}$$

for $i = 0, 1, \ldots, n - 2$ and $k = 0, 1, \ldots, (l - t)/(n - 1) - 1$,

$$E(S_{m_{t-i}}) = \{(\infty, i + rp) \mid r = 0, 1, \ldots, m_t - 1 \}$$

for $i = 0, 1, \ldots, p - 1$,

$$E(S_{m_{t-p-(n-1)/2-k(n-1)-i}}) = \{(i, i + r + \sum_{j=1}^{(l-t)/(n-1)} m_{t-(n-1)/2-(j-1)(n-1)} + \sum_{j=1}^{k} m_{t-p-(n-1)/2-(j-1)(n-1)}) \mid r = 1, 2, \ldots, m_{t-p-(n-1)/2-k(n-1)} \}$$

for $i = 0, 1, \ldots, n - 2$ and $k = 0, 1, \ldots, (t - p - (n - 1)/2)/(n - 1) - 1$.

In each case, the given stars form a rotational decomposition of $K_n$.  

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References


