Packing the Complete Bipartite Graph with Hexagons

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Dedicated to Jimmy Nanney of Auburn University in Montgomery on the event of his retirement (May 2005).

Abstract. Let $K_{m,n}$ denote the complete bipartite graph on $m+n$ vertices with partite sets of cardinalities $m$ and $n$. We give necessary and sufficient conditions for the existence of a 6-cycle packing of $K_{m,n}$.

1. Introduction

A decomposition of a simple graph $G$ into isomorphic copies of a graph $g$ is a set \( \{g_1, g_2, \ldots, g_n\} \) where $g_i \cong g$ and $V(g_i) \subset V(G)$ for all $i$, $E(g_i) \cap E(g_j) = \emptyset$ for $i \neq j$, and $\bigcup_{i=1}^{n} E(g_i) = E(G)$, where $V(G)$ is the vertex set of graph $G$ and $E(G)$ is the edge set of graph $G$. We will refer to such a decomposition as a “$g$ decomposition of $G$.” In the event that a $g$ decomposition of $G$ does not exist, we can ask the question “How close can we get to a $g$ decomposition of $G$?”

A maximal packing of a simple graph $G$ with isomorphic copies of a graph $g$ is a set $\{g_1, g_2, \ldots, g_n\}$ where $g_i \cong g$ and $V(g_i) \subset V(G)$ for all $i$, $E(g_i) \cap E(g_j) = \emptyset$ for $i \neq j$, $\bigcup_{i=1}^{n} g_i \subset G$, and $|E(G) \setminus \bigcup_{i=1}^{n} E(g_i)|$ is minimal.

The set of edges for the leaf, $L$, of the packing is $E(L) = E(G) \setminus \bigcup_{i=1}^{n} E(g_i)$. Packings of complete graphs have been studied, for example, for the graph $g$ a 3-cycle [4], a 4-cycle [5], $K_4$ [1], and a 6-cycle [2, 3].

Let $K_{m,n}$ denote the complete graph on $m+n$ vertices with partite sets of cardinalities $m$ and $n$. Throughout this paper, unless noted otherwise, we denote the partite sets as $V_m$ and $V_n$, where $V_m = \{1_1, 2_1, \ldots, m_1\}$ and $V_n = \{1_2, 2_2, \ldots, n_2\}$. We denote the 6-cycle, $C_6$ or “hexagon,” with
edge set \(\{(a, b), (b, c), (c, d), (d, e), (e, f), (f, a)\}\) as \([a, b, c, d, e, f]\) (and analogously for other length cycles). The purpose of this paper is to give necessary and sufficient conditions for a maximal packing of \(K_{m,n}\) with hexagons.

Conditions for a hexagon decomposition of \(K_{m,n}\) were given by Sotteau [6]:

**Theorem 1.1** The complete bipartite graph \(K_{m,n}\) can be decomposed into hexagons if and only if \(n \equiv 0 \pmod{6}\) and \(n \equiv 0 \pmod{2}\), \(n \geq 4\).

## 2. The Packing Results

We now consider hexagon packings of \(K_{m,n}\).

**Lemma 2.1** A hexagon decomposition of \(K_{n,n} \setminus M\), where \(M\) is a perfect matching of \(K_{n,n}\), exists if and only if \(n \equiv 1\) or \(3 \pmod{6}\).

**Proof.** First we need \(|E(K_{n,n} \setminus M)| = n^2 - n \equiv 0 \pmod{6}\), so \(n \equiv 0\) or \(1\) (mod 3) is necessary. Since each vertex of a hexagon is of even degree and each vertex of \(K_{n,n} \setminus M\) has degree \(n - 1\), we need \(n \equiv 1\) or \(3 \pmod{6}\) is necessary. In this lemma, we assume the vertex set of \(K_{n,n}\) has partite sets \(\{0, 1, 2, \ldots, (n-1)\}\) and \(\{0, 2, 4, \ldots, (n-2)\}\).

We now consider cases. In each case, the vertex labels are reduced modulo \(n\) and the collection of hexagons forms a decomposition of \(K_{n,n}\).

**Case 1.** Suppose \(n \equiv 1 \pmod{12}\), say \(n = 12k + 1\). Consider the hexagons:
\[
\{[i_1, (12j+i), 2], (12k+i) 1, (12j+1+i) 2, (12k-1+i) 1, (12j+4+i) 2, [i_1, (12j+5+i) 2], (12k-1+i) 1, (12j+7+i) 2, (12k+i) 1, (12j+10+i) 2 \mid i = 0, 1, \ldots, 12k; j = 0, 1, \ldots, k-1\}.
\]
In this case, \(E(M) = \{(i_1, (12k+i) 2) \mid i = 0, 1, \ldots, 12k\}\).

**Case 2.** Suppose \(n \equiv 7 \pmod{12}\), say \(n = 12k + 7\). Consider the hexagons:
\[
\{[i_1, (12j+i) 2], (12k+6+i) 1, (12j+1+i) 2, (12k+5+i) 1, (12j+4+i) 2, [i_1, (12j+5+i) 2], (12k+6+i) 1, (12j+7+i) 2, (12k+i) 1, (12j+10+i) 2 \mid i = 0, 1, \ldots, 12k+6; j = 0, 1, \ldots, k-1\} \cup \{[i_1, (12j+i) 2], (12k+6+i) 1, (12j+4+i) 2, [i_1, (12j+5+i) 2], (12k+i) 1, (12j+7+i) 2, (12k+6+i) 1, (12j+10+i) 2 \mid i = 0, 1, \ldots, 12k+6; j = 0, 1, \ldots, k-1\}.
\]
In this case, \(E(M) = \{(i_1, (12k+5+i) 2) \mid i = 0, 1, \ldots, 12k+6\}\).

**Case 3.** Suppose \(n \equiv 3 \pmod{36}\), say \(n = 36k + 3\). Consider the hexagons:
\[
\{[i_1, (12j+i) 2], (36k+2+i) 1, (12j+1+i) 2, (36k+1+i) 1, (12j+4+i) 2, [i_1, (12j+5+i) 2], (36k+1+i) 1, (12j+7+i) 2, (36k+2+i) 1, (12j+10+i) 2 \mid i = 0, 1, \ldots, 36k+2; j = 0, 1, \ldots, k-1\} \cup \{[i_1, (12j+14k+14+i) 2], (36k+2+i) 1, (12j+14k+14+i) 2, (36k+2+i) 1, (12j+14k+14+i) 2, (36k+2+i) 1, (12j+14k+14+i) 2, (36k+2+i) 1, (12j+14k+14+i) 2 \mid i = 0, 1, \ldots, 36k+2; j = 0, 1, \ldots, k-2\} \cup \{[i_1, (12j+24k+14+i) 2], (36k+2+i) 1, (12j+24k+14+i) 2, (36k+2+i) 1, (12j+24k+14+i) 2, (36k+2+i) 1, (12j+24k+14+i) 2, (36k+2+i) 1, (12j+24k+14+i) 2 \mid i = 0, 1, \ldots, 36k+2; j = 0, 1, \ldots, k-2\}.
\]
\[i = 0, 1, \ldots, 36k+2; j = 0, 1, \ldots, k-2 \cup \{[i_1, (12k+i)_2, (36k+1+i)_1, (12k+2+i)_2, (36k+2+i)_1, (12k+5+i)_2, (36k+7+i)_1, (12k+10+i)_2] | i = 0, 1, \ldots, 36k+2 \} \cup \{[i_1, (12k+i)_2, (36k+2+i)_1, (12k+5+i)_2, (36k+7+i)_1, (12k+10+i)_2] | i = 0, 1, \ldots, 36k+2 \} \cup \{[i_1, (12k+i)_2, (36k+1+i)_1, (12k+5+i)_2, (36k+7+i)_1, (12k+10+i)_2] | i = 0, 1, \ldots, 36k+2 \}

In this case, \(E(M) = \{(i_1, (36k+2+i)_2) | i = 0, 1, \ldots, 36k+2 \} \).

**Case 4.** Suppose \(n \equiv 9 \pmod{36}\), say \(n = 36k+9\). Consider the hexagons:
\[\{[i_1, (12j+i)_2, (36k+8+i)_1, (12j+1+i)_2, (36k+7+i)_1, (12j+4+i)_2] | i = 0, 1, \ldots, 36k+8; j = 0, 1, \ldots, k-1 \} \cup \{[i_1, (12j+2k+7+i)_2, (36k+8+i)_1, (12j+10+i)_2] | i = 0, 1, \ldots, 36k+8; j = 0, 1, \ldots, k-1 \} \cup \{[i_1, (12j+2k+11+i)_2, (36k+8+i)_1, (12j+12+i)_2] | i = 0, 1, \ldots, 36k+8; j = 0, 1, \ldots, k-1 \}
\]

In this case, \(E(M) = \{(i_1, (36k+7+i)_2) | i = 0, 1, \ldots, 36k+8 \} \).

**Case 5.** Suppose \(n \equiv 15 \pmod{36}\), say \(n = 36k+15\). Consider the hexagons:
\[\{[i_1, (12j+i)_2, (36k+14+i)_1, (12j+1+i)_2, (36k+13+i)_1, (12j+4+i)_2] | i = 0, 1, \ldots, 36k+14; j = 0, 1, \ldots, k-1 \} \cup \{[i_1, (12j+12k+7+i)_2, (36k+14+i)_1, (12j+12k+11+i)_2] | i = 0, 1, \ldots, 36k+14; j = 0, 1, \ldots, k-1 \}\]

In this case, \(E(M) = \{(i_1, (36k+14+i)_2) | i = 0, 1, \ldots, 36k+14 \} \).

**Case 6.** Suppose \(n \equiv 21 \pmod{36}\), say \(n = 36k+21\). Consider the hexagons:
\[\{[i_1, (12j+i)_2, (36k+20+i)_1, (12j+1+i)_2, (36k+19+i)_1, (12j+4+i)_2] | i = 0, 1, \ldots, 36k+20; j = 0, 1, \ldots, k-1 \} \cup \{[i_1, (12j+12k+19+i)_2, (36k+20+i)_1, (12j+12k+23+i)_2] | i = 0, 1, \ldots, 36k+20; j = 0, 1, \ldots, k-1 \}
\]

In this case, \(E(M) = \{(i_1, (36k+20+i)_2) | i = 0, 1, \ldots, 36k+20 \} \).
\( \{ (i_1, (36k + 19 + i)_2) \mid i = 0, 1, \ldots, k \} \}

In this case, \( E(M) = \{ (i_1, (36k + 19 + i)_2) \mid i = 0, 1, \ldots, 36k + 20 \} \).

Case 7. Suppose \( n \equiv 27 \pmod{36} \), say \( n = 36k + 27 \). Consider the hexagons: \( \{ (i_1, (12j + i)_2, (36k + 26 + i)_1, (12j + 1 + i)_2, (36k + 25 + i)_1, (12j + 4 + i)_2) \mid i = 0, 1, \ldots, k - 1 \} \).

In this case, \( E(M) = \{ (i_1, (36k + 25 + i)_1, (12j + 4 + i)_2) \} \).

Case 8. Suppose \( n \equiv 33 \pmod{36} \), say \( n = 36k + 33 \). Consider the hexagons: \( \{ (i_1, (12j + i)_2, (36k + 32 + i)_1, (12j + 1 + i)_2, (36k + 31 + i)_1, (12j + 4 + i)_2) \mid i = 0, 1, \ldots, k - 1 \} \).

In this case, \( E(M) = \{ (i_1, (36k + 32 + i)_1, (12j + 4 + i)_2) \} \).
Lemma 2.2 A maximal hexagon packing of $K_{m,n}$ where $m$ is even and $n$ is odd ($m \geq 4, n \geq 3$) has a leave $L$ satisfying $|E(L)| = m + k$ where $k$ is the smallest nonnegative integer such that $|E(K_{m,n})| - (m + k) \equiv 0 \pmod{6}$.

Proof. Since each vertex of $V_m$ is of odd degree in $K_{m,n}$, in the leave of a packing each of these vertices will be of odd degree. Therefore in a packing of $K_{m,n}$ with leave $L$, it is necessary that $|E(L)| \geq m$. Since $K_{m,n}$ is a union of $L$ and a collection of hexagons, then $|E(K_{m,n})| \equiv |E(L)| \pmod{6}$. So in a maximal packing, it is necessary that $|E(L)| = m + k$ where $k$ is as described. We now establish sufficiency.

Case 1. Suppose $m \equiv 0 \pmod{6}$ and $n \equiv 1 \pmod{2}$. Now $K_{m,n} = K_{m,n-3} \cup \frac{m}{3} \times (K_{3,3} \setminus M) \cup \frac{n}{2} \times M$ where the partite sets of $K_{m,n-3}$ are $V_m$ and $V_n \setminus \{1_2, 2_2, 3_2\}$, the partite sets of the $i$th $K_{3,3} \setminus M$ are $\{(3i - 2)_1, (3i - 1)_1, (3i)_1\}$ and $\{1_2, 2_2, 3_2\}$, and $M$ is a perfect matching of $K_{3,3}$. Now $K_{m,n-3}$ can be decomposed into hexagons by Theorem 1.1 and each $K_{3,3} \setminus M$ can be decomposed into hexagons by Lemma 2.1. Therefore a maximal packing of $K_{m,n}$ will have a leave $L$ where $|E(L)| = m$.

Case 2. Suppose $m \equiv 0 \pmod{2}$, $m \geq 4$, and $n \equiv 1 \pmod{6}$. Now $K_{m,n} = K_{m,n-1} \cup S_m$ where the partite sets of $K_{m,n-1}$ are $V_m$ and $V_n \setminus \{1_2\}$, and the edge set of $S_m$ is $\{(i_1, 1_2) \mid i = 1, 2, \ldots, m\}$. Now $K_{m,n-1}$ can be decomposed into hexagons by Theorem 1.1. Therefore a maximal packing of $K_{m,n}$ will have a leave $L$ where $|E(L)| = m$.

Case 3. Suppose $m \equiv 2 \pmod{6}$, $m \geq 4$, and $n \equiv 3 \pmod{6}$. Now $|E(K_{m,n})| - m \equiv 4 \pmod{6}$, so it is necessary that a packing have a leave $L$ with $|E(L)| \geq m + 4$. Now $K_{m,n} = K_{m,n-3} \cup (K_{m-8,3} \setminus M) \cup 2 \times C_6 \cup M \cup 12 \times K_2$ where the partite sets of $K_{m,n-3}$ are $V_m$ and $V_n \setminus \{1_2, 2_2, 3_2\}$, the partite sets of $K_{m-8,3} \setminus M$ are $\{9_1, 10_1, \ldots, m_1\}$ and $\{1_2, 2_2, 3_2\}$, $M$ is a collection of $m - 8$ edges of $K_{m-8,3}$ as described in Case 1, 2 $\times C_6 = \{(3_1, 2_2, 4_1, 3_2, 5_1, 1_2), (6_1, 3_2, 8_1, 2_2, 7_1, 1_2)\}$, and $12 \times K_2 = \{(1_1, 1_2), (1_1, 2_2), (1_1, 3_2), (2_1, 1_2), (2_1, 2_2), (2_1, 3_2), (3_1, 3_2), (4_1, 1_2), (5_1, 2_2), (6_1, 2_2), (7_1, 3_2), (8_1, 1_2)\}$. Now $K_{m,n-3}$ can be decomposed into hexagons by Theorem 1.1 and $K_{m-8,3} \setminus M$ can be decomposed into hexagons by Case 1. Therefore a maximal packing of $K_{m,n}$ will have a leave $L$ where $|E(L)| = |E(M)| + |E(12 \times K_2)| = m + 4$.

Case 4. Suppose $m \equiv 2 \pmod{6}$, $m \geq 8$, and $n \equiv 5 \pmod{6}$. Now
\(|E(K_{m,n})| - m \equiv 2 \pmod{6}\), so it is necessary that a packing have a
leave \(L\) with \(|E(L)| \geq m + 2\). Now \(K_{m,n} = K_{m,n-5} \cup (K_{m-8,5} \setminus M) \cup
5 \times C_6 \cup 10 \times K_2\) where the partite sets of \(K_{m,n-5}\) are \(V_m\) and \(V_n \setminus
\{1_2, 2_2, 3_2, 4_2, 5_2\}\), the partite sets of \(K_{m-8,5} \setminus M\) are \(\{9_1, 10_1, \ldots, m_1\}\) and
\(\{1_2, 2_2, 3_2, 4_2, 5_2\}\), \(M\) is a collection of \(m - 8\) edges as described in Case 1, \(5 \times C_6 = \{(1_1, 2_1, 2_2, 3_1, 3_2), [2_1, 3_2, 4_1, 5_2, 3_1, 4_2], [5_1, 1_2, 6_1, 2_2, 7_1, 3_2], [6_1, 3_2, 8_1, 5_2, 7_1, 4_2], [8_1, 1_2, 4_1, 2_1, 1_2, 4_2]\}\), and \(10 \times K_2 = \{(1_1, 5_2), (2_1, 5_2), (3_1, 1_2), (4_1, 4_2), (5_1, 2_2), (5_1, 4_2), (5_1, 5_2), (6_1, 5_2), (7_1, 1_2), (8_1, 2_2)\}\). Now \(K_{m,n-5}\) can be decomposed into hexagons by Theorem 1.1 and \(K_{m-8,5} \setminus M\) can be decomposed into hexagons by Case 1. Therefore a maximal packing of
\(K_{m,n}\) will have a leave \(L\) where \(|E(L)| = |E(M)| + |E(10 \times K_2)| = m + 2\).

Case 5. Suppose \(m \equiv 4 \pmod{6}\) and \(n \equiv 3 \pmod{6}\). As in Case 4, a
packing with leave \(L\) satisfies \(|E(L)| \geq m + 2\). Now \(K_{m,n} = K_{m,n-3} \cup
(K_{n-4,3} \setminus M) \cup C_6 \cup M \cup 6 \times K_2\) where the partite sets of \(K_{m,n-3}\) are \(V_m\) and
\(V_n \setminus \{1_2, 2_2, 3_2\}\), the partite sets of \(K_{n-4,3} \setminus M\) are \(V_m \setminus \{1_1, 2_1, 3_1, 4_1\}\) and
\(\{1_2, 2_2, 3_2\}\), \(M\) is a collection of \(m - 4\) edges of \(K_{n-4,3}\) as described in Case 1, \(C_6 = \{(1_1, 1_2, 2_1, 2_2, 3_1, 3_2)\}\), and \(6 \times K_2 = \{(1_1, 2_2), (2_1, 3_2), (3_1, 1_2), (4_1, 1_2), (4_1, 2_2), (4_1, 3_2)\}\). Now \(K_{m,n-3}\) can be decomposed into hexagons by Theorem 1.1 and \(K_{n-4,3} \setminus M\) can be decomposed into hexagons by Case 1. Therefore a maximal packing of \(K_{m,n}\) will have a leave \(L\) were
\(|E(L)| = |E(M)| + |E(6 \times K_2)| = m + 2\).

Case 6. Suppose \(m \equiv 4 \pmod{6}\) and \(n \equiv 5 \pmod{6}\). As in Case 3, a
packing with leave \(L\) satisfies \(|E(L)| \geq m + 4\). Now \(K_{m,n} = K_{m,n-5} \cup (K_{m-4,5} \setminus
M) \cup 2 \times C_6 \cup M \cup 8 \times K_2\) where the partite sets of \(K_{m,n-5}\) are \(V_m\) and \(V_n \setminus
\{1_2, 2_2, 3_2, 4_2, 5_2\}\), the partite sets of \(K_{m-4,5} \setminus M\) are \(V_m \setminus \{1_1, 2_1, 3_1, 4_1\}\) and
\(\{1_2, 2_2, 3_2, 4_2, 5_2\}\), \(M\) is a collection of \(m - 4\) edges of \(K_{m-4,5}\) as described in Case 1, \(2 \times C_6 = \{(1_1, 1_2, 2_1, 2_2, 3_1, 3_2), [2_1, 3_2, 4_1, 5_2, 3_1, 4_2]\}\) and \(8 \times K_2 = \{(1_1, 2_2), (1_1, 4_2), (1_1, 5_2), (2_1, 5_2), (3_1, 1_2), (4_1, 1_2), (4_1, 2_2), (4_1, 4_2)\}\). Now \(K_{m,n-5}\) can be decomposed into hexagons by Theorem 1.1 and \(K_{m-4,5} \setminus M\) can be decomposed into hexagons by Case 1. Therefore a maximal packing of \(K_{m,n}\) will have a leave \(L\) where \(|E(L)| = |E(M)| + |E(8 \times K_2)| = m + 4\).

Lemma 2.3 A maximal hexagon packing of \(K_{m,n}\) where \(m\) and \(n\) are both
odd, \(m \geq n \geq 3\), has a leave \(L\) satisfying \(|E(L)| = m + k\) where \(k\) is the
smallest nonnegative integer such that \(|E(K_{m,n})| - (m + k) \equiv 0 \pmod{6}\).

Proof. The necessary conditions follow as in Lemma 2.2. We now establish
sufficiency.

Case 1. Suppose \(m \equiv 1 \pmod{6}\), \(n \equiv 1 \pmod{6}\), and \(m \geq n\). Now
\(K_{m,n} = (K_{n,n} \setminus M_1) \cup (K_{m-n,n} \setminus M_2) \cup M_1 \cup M_2\) where the partite sets of
\(K_{n,n} \setminus M_1\) are \(\{1_1, 2_1, \ldots, n_1\}\) and \(V_n, M_1\) is a perfect matching of \(K_{n,n}\),
the partite sets of \(K_{m-n,n} \setminus M_2\) are \(\{(n+1)_1, (n+2)_1, \ldots, m_1\}\) and \(V_n,\)
and $M_2$ is a set of $m-n$ edges of $K_{m-n,n}$ as described in Case 1 of Lemma 2.2. Now $K_{n,n} \setminus M_1$ can be decomposed into hexagons by Lemma 2.1 and $K_{m-n,n} \setminus M_2$ can be decomposed into hexagons by Lemma 2.2 Case 1. Therefore a maximal packing of $K_{m,n}$ will have a leave $L$ where $|E(L)| = m$.

**Case 2.** Suppose $m \equiv 1 \pmod{6}$, $n \equiv 3 \pmod{6}$, and $m > n$. As in Case 4 of Lemma 2.2, a packing with leave $L$ satisfies $|E(L)| \geq m + 2$. Now $K_{m,n} = (K_{n,n} \setminus M_1) \cup (K_{m-n,n} \setminus M_2) \cup M_1 \cup M_2$ where the partite sets of $K_{n,n} \setminus M_1$ are $\{1, 2, \ldots, n_1\}$ and $V_n$, $M_1$ is a perfect matching of $K_{n,n}$, the partite sets of $K_{m-n,n} \setminus M_2$ are $\{(n+1)_1, (n+2)_1, \ldots, m_1\}$ and $V_n$, and $M_2$ is a set of $m-n+2$ edges of $K_{m-n,n}$ as described in Case 5 of Lemma 2.2. Now $K_{n,n} \setminus M_1$ can be decomposed into hexagons by Lemma 2.1 and $K_{m-n,n} \setminus M_2$ can be decomposed into hexagons by Lemma 2.2 Case 5. Therefore a maximal packing of $K_{m,n}$ will have a leave $L$ where $|E(L)| = m + 2$.

**Case 3.** Suppose $m \equiv 1 \pmod{6}$, $n \equiv 5 \pmod{6}$, and $m > n$. As in Case 3 of Lemma 2.2, a packing with leave $L$ satisfies $|E(L)| \geq m + 4$. Now $K_{m,n} = (K_{n-4,n-4} \setminus M_1) \cup (K_{m-n+4,n} \setminus M_2) \cup (K_{m,4} \setminus M_3) \cup M_1 \cup M_2 \cup M_3$ where the partite sets of $K_{n-4,n-4} \setminus M_1$ are $\{1, 2, \ldots, (n-4)_1\}$ and $V_n \setminus \{(n-3)_2, (n-2)_2, (n-1)_2, n_2\}$, $M_1$ is a perfect matching of $K_{n-4,n-4}$, the partite sets of $K_{m-n+4,n} \setminus M_2$ are $\{(n-3)_1, (n-2)_1, \ldots, m_1\}$ and $V_n$, and $M_2$ is a set of $m-n+4$ edges of $K_{m-n+4,n}$ as described in Case 1 of Lemma 2.2, the partite sets of $K_{m,4} \setminus M_3$ are $V_m$ and $\{(n-3)_2, (n-2)_2, (n-1)_2, n_2\}$, and $M_3$ is a set of 4 edges as described in Case 1 of Lemma 2.2. Now $K_{n-4,n-4} \setminus M_1$ can be decomposed into hexagons by Lemma 2.1, and $K_{m-n+4,n} \setminus M_2$ and $K_{m,4} \setminus M_3$ can be decomposed into hexagons by Lemma 2.2 Case 1. Therefore a maximal packing of $K_{m,n}$ will have a leave $L$ where $|E(L)| = m + 4$.

**Case 4.** Suppose $m \equiv 3 \pmod{6}$, $n \equiv 1 \pmod{6}$, and $m > n$. This case follows the same as Case 1, but in this case $K_{m-n,n} \setminus M_2$ can be decomposed into hexagons by Lemma 2.2 Case 2. Again, the leave $L$ in a maximal packing satisfies $|E(L)| = m$.

**Case 5.** Suppose $m \equiv 3 \pmod{6}$, $n \equiv 3 \pmod{6}$, and $m \geq n$. This case follows the same as Case 1. Again, the leave $L$ in a maximal packing satisfies $|E(L)| = m$.

**Case 6.** Suppose $m \equiv 3 \pmod{6}$, $n \equiv 5 \pmod{6}$, and $m > n$. Now $K_{m,n} = K_{n-5,4} \cup K_{m-n+4,n-5} \cup K_{m-n+4,4} \cup (K_{n-4,n-4} \setminus M) \cup 6 \times C_6 \cup 8 \times K_2 \cup S_{m-n-4}$ where the partite sets of $K_{n-5,4}$ are $\{1, 2, \ldots, (n-5)_1\}$ and $\{(m-3)_2, (m-2)_2, (m-1)_2, m_2\}$, the partite sets of $K_{m-n+4,n-5}$ are $\{(n-3)_1, (n-2)_1, \ldots, m_1\}$ and $V_n \setminus \{(n-4)_2, (n-3)_2, (n-2)_2, (n-1)_2, n_2\}$, the partite sets of $K_{m-n+4,4}$ are $\{(n+5)_1, (n+6)_1, \ldots, m_1\}$ and $\{(n-3)_2, (n-2)_2, (n-1)_2, n_2\}$, the partite sets of $K_{n-4,n-4} \setminus M$ are $\{1, 2, \ldots, (n-4)_1\}$ and $V_n \setminus \{(n-3)_2, (n-2)_2, (n-1)_2, n_2\}$, $M$ is a perfect matching of $K_{n-4,n-4}$ containing the edge $((n-4)_1, (n-4)_2)$, $6 \times$
\[ C_6 = \{ ((n-3)_1, (n-3)_2, (n-4)_1, (n-1)_2, n_1, (n-2)_2), (n-3)_2, n_1, (n-4)_2, (n-2)_1, n_2), \ldots \} \]

Now \( K_{n-5,4}, K_{m-n-4, n-5}, \) and \( K_{m-n-4,4} \) can be decomposed into hexagons by Theorem 1.1, and \( K_{n-4,n-4} \) can be decomposed into hexagons by Lemma 2.1. Therefore a maximal packing of \( K_{m,n} \) will have a leave \( L \) where \(|E(L)| = |E(M)| + |E(8 \times K_2)| + |E(S_{m-n-4})| = m\).

**Case 7.** Suppose \( m \equiv 5 \pmod{6}, n \equiv 1 \pmod{6}, \) and \( m > n \). This case follows the same as Case 1, but in this case \( K_{m-n,n} \setminus M_2 \) can be decomposed into hexagons by Lemma 2.2 Case 2. Again, the leave \( L \) in a maximal packing satisfies \(|E(L)| = m\).

**Case 8.** Suppose \( m \equiv 5 \pmod{6}, n \equiv 3 \pmod{6}, \) and \( m \geq n \). As in Case 3 of Lemma 2.2, a packing with leave \( L \) satisfies \(|E(L)| \geq m + 4\). Now \( K_{m,n} = (K_{m,n} \setminus M_1) \cup (K_{m-n,n} \setminus M_2) \cup M_1 \cup M_2 \) where the partite sets of \( K_{m,n} \setminus M_1 \) are \( \{1, 2, \ldots, n_1\} \) and \( V_n, M_1 \) is a perfect matching of \( K_{m,n} \), the partite sets of \( K_{m-n,n} \setminus M_2 \) are \( \{(n+1)_1, (n+2)_1, \ldots, m_1\} \) and \( v_n \), and \( M_2 \) is a set of \( m - n + 4 \) edges of \( K_{m-n,n} \) as described in Case 3 of Lemma 2.2. Now \( K_{m,n} \setminus M_1 \) can be decomposed into hexagons by Lemma 2.1 and \( K_{m-n,n} \setminus M_2 \) can be decomposed into hexagons by Lemma 2.2 Case 3. Therefore a maximal packing of \( K_{m,n} \) will have a leave \( L \) where \(|E(L)| = |E(M_1)| + |E(M_2)| = m + 4\).

**Case 9.** Suppose \( m \equiv 5 \pmod{6}, n \equiv 5 \pmod{6}, \) and \( m \geq n \). As in Case 4 of Lemma 2.2, a packing with leave \( L \) satisfies \(|E(L)| \geq m + 2\). Now \( K_{m,n} = K_{4,n-5} \cup K_{n-5,4} \cup (K_{n-4,n-4} \setminus M_1) \cup (K_{m-n,n} \setminus M_2) \cup 3 \times C_6 \cup 6 \times K_2 \cup M_1 \cup M_2 \) where the partite sets of \( K_{4,n-5} \) are \( \{1, 2, 3, 4, 1\} \) and \( V_n \setminus \{1, 2, 3, 4, 5\} \), the partite sets of \( K_{n-5,4} \) are \( V_n \setminus \{1, 2, 3, 4, 5, 1\} \) and \( \{2, 3, 2, 3\} \), the partite sets of \( K_{n-4,n-4} \setminus M_1 \) are \( \{5, 1, 2, \ldots, m_1\} \) and \( V_n \setminus \{2, 3, 2, 4\} \). \( M_1 \) is a perfect matching of \( K_{n-4,n-4} \) which contains the edge \((5_1, 5_2)\), the partite sets of \( K_{m-n,n} \setminus M_2 \) are \( \{(n+1)_1, (n+2)_1, \ldots, m_1\} \) and \( V_n \), \( M_2 \) is a set of \( m - n \) edges as described in Case 1 of Lemma 2.2, \( 3 \times C_6 = \{[1, 2, 2, 1, 2, 5, 1, 4, 2], [3, 2, 5, 1, 2, 1, 4, 1, 5, 2], [1, 3, 2, 3, 4, 2, 5, 1, 2, 1, 5, 2] \}\), and \( 6 \times \{([1, 2, 2, 1, 2, 3, 1, 2], [4, 1, 2, 2], [4, 1, 3, 2], [4, 1, 2, 2]) \). Now \( K_{4,n-5} \) and \( K_{n-5,4} \) can be decomposed into hexagons by Theorem 1.1, \( K_{n-4,n-4} \setminus M_1 \) can be decomposed into hexagons by Lemma 2.1, and \( K_{m-n,n} \setminus M_2 \) can be decomposed into hexagons by Lemma 2.2 Case 1. Therefore a maximal packing of \( K_{m,n} \) will have a leave \( L \) where \(|E(L)| = |E(M_1)| + |E(M_2)| + |E(6 \times K_2)| = m + 2\).
Lemma 2.4 A maximal hexagon packing of $K_{m,n}$ where $m$ and $n$ are even, $m, n \geq 4$, has a leave $L$ satisfying:
(1) $|E(L)| = 0$ when $m \equiv 0 \pmod{6}$,
(2) $|E(L)| = 4$ when $m \equiv n \equiv 2 \pmod{6}$ or $m \equiv n \equiv 4 \pmod{6}$, and
(3) $|E(L)| = 8$ when $m \equiv 2 \pmod{6}$ and $n \equiv 4 \pmod{6}$.

Proof. We consider cases.
Case 1. Suppose $m \equiv 0 \pmod{6}$, $n \equiv 0 \pmod{2}$, and $n \geq 4$. Then $K_{m,n}$ can be decomposed into hexagons by Theorem 1.1 and in a maximal packing, $|E(L)| = 0$.

Case 2. Suppose $m \equiv n \equiv 2 \pmod{6}$, $m, n \geq 4$, Now $|E(K_{m,n})| \equiv 4 \pmod{6}$, so it is necessary that a packing have leave $L$ with $|E(L)| \geq 4$. Now $K_{m,n} = K_{m-8,n} \cup K_{8,n-8} \cup 10 \times C_6 \cup C_4$ where the partite sets of $K_{m-8,n}$ are $\{9_1, 10_1, \ldots, m_1\}$ and $V_n$, the partite sets of $K_{8,n-8}$ are $\{1_1, 2_1, \ldots, 8_1\}$ and $\{9_2, 10_2, \ldots, n_2\}$, $10 \times C_6 = \{[2_1, 2_2, 4_1, 8_2, 1_1, 3_2], [3_1, 3_2, 5_1, 1_2, 2_1, 4_2], [4_1, 4_2, 6_1, 2_2, 3_1, 5_2], [5_1, 5_2, 7_1, 3_2, 4_1, 6_2], [7_1, 7_2, 1_1, 5_2, 6_1, 8_2], [8_1, 8_2, 2_1, 6_2, 7_1, 1_2], [9_1, 3_2, 6_1, 1_2, 4_1, 7_2], [6_1, 6_2, 1_1, 4_2, 5_1, 7_2], [1_1, 1_2, 3_1, 8_2, 5_1, 3_2], [2_1, 5_2, 8_1, 6_2, 3_1, 7_2]\}$, and $C_4 = \{7_1, 2_2, 8_1, 4_2\}$. Now $K_{m-8,n}$ and $K_{8,n-8}$ can be decomposed into hexagons by Theorem 1.1. Therefore a maximal packing of $K_{m,n}$ will have a leave $L$ where $|E(L)| = |E(C_4)| = 4$.

Case 3. Suppose $m \equiv n \equiv 4 \pmod{6}$, As in Case 2, a packing with leave $L$ satisfies $|E(L)| \geq 4$. Now $K_{m,n} = K_{m-4,n} \cup K_{4,n-4} \cup 2 \times C_6 \cup C_4$ where the partite sets of $K_{m-4,n}$ are $V_m \setminus \{1_1, 2_1, 3_1, 4_1\}$ and $V_n$, the partite sets of $K_{4,n-4}$ are $\{1_1, 2_1, 3_1, 4_1\}$ and $\{1_2, 2_2, 3_2, 4_2\}$, $2 \times C_6 = \{[1_1, 1_2, 2_1, 2_2, 3_1, 3_2], [4_1, 4_2, 6_1, 2_2, 3_1, 5_2], [5_1, 5_2, 7_1, 3_2, 4_1, 6_2], [7_1, 7_2, 1_1, 5_2, 6_1, 8_2], [8_1, 8_2, 2_1, 6_2, 7_1, 1_2], [9_1, 3_2, 6_1, 1_2, 4_1, 7_2], [6_1, 6_2, 1_1, 4_2, 5_1, 7_2], [1_1, 1_2, 3_1, 8_2, 5_1, 3_2], [2_1, 5_2, 8_1, 6_2, 3_1, 7_2]\}$, and $C_4 = \{1_1, 2_2, 4_1, 4_2\}$. Now $K_{m-4,n}$ and $K_{4,n-4}$ can be decomposed into hexagons by Theorem 1.1. Therefore a maximal packing of $K_{m,n}$ will have a leave $L$ where $|E(L)| = |E(C_4)| = 4$.

Case 4. Suppose $m \equiv 2 \pmod{6}$, $m \geq 8$, and $n \equiv 4 \pmod{6}$. Now $|E(K_{m,n})| \equiv 2 \pmod{6}$, so it is necessary that a packing have a leave $L$ with $|E(L)| \geq 2$. Now each vertex of $K_{m,n}$ is of even degree and each vertex of $C_6$ is of even degree, so for any packing, the vertices of $L$ must be of even degree. However, we cannot have this property when $|E(L)| = 2$. So it is necessary that a packing have a leave $L$ with $|E(L)| \geq 8$. Now $K_{m,n} = K_{8,n-4} \cup K_{m-8,n} \cup 4 \times C_6 \cup 2 \times C_4$ where the partite sets of $K_{8,n-4}$ are $\{1_1, 2_1, \ldots, 8_1\}$ and $V_n \setminus \{1_2, 2_2, 3_2, 4_2\}$, the partite sets of $K_{m-8,n}$ are $\{9_1, 10_1, \ldots, m_1\}$ and $V_n, 4 \times C_6 = \{[1_1, 1_2, 2_1, 2_2, 3_1, 3_2], [4_1, 4_2, 6_1, 2_2, 3_1, 5_2], [5_1, 5_2, 7_1, 3_2, 4_1, 6_2], [7_1, 7_2, 1_1, 5_2, 6_1, 8_2], [8_1, 8_2, 2_1, 6_2, 7_1, 1_2], [9_1, 3_2, 6_1, 1_2, 4_1, 7_2], [6_1, 6_2, 1_1, 4_2, 5_1, 7_2], [1_1, 1_2, 3_1, 8_2, 5_1, 3_2], [2_1, 5_2, 8_1, 6_2, 3_1, 7_2]\}$, and $2 \times C_4 = \{[1_1, 2_2, 4_1, 4_2], [2_2, 5_1, 4_2, 8_1]\}$. Now $K_{8,n-4}$ and $K_{m-8,n}$ can be decomposed into hexagons by Theorem 1.1. Therefore a maximal packing of $K_{m,n}$ will have a leave $L$ where $|E(L)| = |E(2 \times C_4)| = 8$.
Lemmas 2.2–2.4 combine to give our main result.

**Theorem 2.1** A maximal hexagon packing of $K_{m,n}$ with leave $L$ satisfies

1. when $m \equiv 0 \pmod{2}$ and $n \equiv 1 \pmod{2}$, $|E(L)| = m + k$ where $k$ is the smallest nonnegative integer such that $|E(K_{m,n})| - (m + k) \equiv 0 \pmod{6}$,
2. when $m \equiv n \equiv 1 \pmod{2}$ and $m \geq n$, $|E(L)| = m + k$ where $k$ is the smallest nonnegative integer such that $|E(K_{m,n})| - (m + k) \equiv 0 \pmod{6}$,
3. when $m \equiv 0 \pmod{6}$ and $n \equiv 0 \pmod{2}$, $|E(L)| = 0$,
4. when $m \equiv n \equiv 2 \pmod{6}$ or $m \equiv n \equiv 4 \pmod{6}$, then $|E(L)| = 4$, and
5. when $m \equiv 2 \pmod{6}$ and $n \equiv 4 \pmod{6}$, then $|E(L)| = 8$.

**References**


[6] D. Sotteau, Decompositions of $K_{m,n}$ ($K_{m,n}^*$) into Cycles (Circuits) of Length $2k$, *Journal of Combinatorial Theory, Series B*, 30 (1981), 75-81.