Packing the Complete Bipartite Graph with Hexagons

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Dedicated to Jimmy Nanney of Auburn University in Montgomery on the event of his retirement (May 2005).

Abstract. Let $K_{m,n}$ denote the complete bipartite graph on $m+n$ vertices with partite sets of cardinalities m and n. We give necessary and sufficient conditions for the existence of a 6-cycle *packing* of $K_{m,n}$.

1. Introduction

A *decomposition* of a simple graph G into isomorphic copies of a graph g is a set $\{g_1, g_2, \ldots, g_n\}$ where $g_i \cong g$ and $V(g_i) \subset V(G)$ for all i, $E(g_i) \bigcap E(g_j) = \emptyset$ for $i \neq j$, and $\bigcup_{i=1}^{n} E(g_i) = E(G)$, where $V(G)$ is the vertex set of graph G and $E(G)$ is the edge set of graph G. We will refer to such a decomposition as a "g decomposition of G." In the event that a g decomposition of G does not exist, we can ask the question "How close can we get to a g decomposition of G?"

A *maximal packing* of a simple graph G with isomorphic copies of a graph g is a set $\{g_1, g_2, \ldots, g_n\}$ where $g_i \cong g$ and $V(g_i) \subset V(G)$ for all i, $E(g_i) \bigcap E(g_j) = \emptyset$ for $i \neq j$, $\bigcup_{i=1}^{n} g_i \subset G$, and $|E(G) \setminus \bigcup_{i=1}^{n} E(g_i)|$ is minimal. The set of edges for the *leave*, L, of the packing is $E(L) = E(G) \setminus \bigcup_{i=1}^{n} E(g_i)$. Packings of complete graphs have been studied, for example, for the graph g a 3-cycle [4], a 4-cycle [5], K_4 [1], and a 6-cycle [2, 3].

Let $K_{m,n}$ denote the complete graph on $m+n$ vertices with partite sets of cardinalities m and n. Throughout this paper, unless noted otherwise, we denote the partite sets as V_m and V_n, where $V_m = \{1_1, 2_1, \ldots, m_1\}$ and $V_n = \{1_2, 2_2, \ldots, n_2\}$. We denote the 6-cycle, C_6 or "hexagon," with edge set $\{(a,b),(b,c),(c,d),(d,e),(e,f),(f,a)\}$ as $[a,b,c,d,e,f]$ (and analogously for other length cycles). The purpose of this paper is to give necessary and sufficient conditions for a maximal packing of $K_{m,n}$ with hexagons.

Conditions for a hexagon decomposition of $K_{m,n}$ were given by Sotteau [6]:

Theorem 1.1 *The complete bipartite graph $K_{m,n}$ can be decomposed into hexagons if and only if $m \equiv 0 \pmod 6$ and $n \equiv 0 \pmod 2$, $n \geq 4$.*

2. The Packing Results

We now consider hexagon packings of $K_{m,n}$.

Lemma 2.1 *A hexagon decomposition of $K_{n,n} \setminus M$, where M is a perfect matching of $K_{n,n}$, exists if and only if $n \equiv 1$ or $3 \pmod 6$.*

Proof. First we need $|E(K_{n,n} \setminus M)| = n^2 - n \equiv 0 \pmod 6$, so $n \equiv 0$ or $1 \pmod 3$ is necessary. Since each vertex of a hexagon is of even degree and each vertex of $K_{n,n} \setminus M$ has degree $n-1$, we need n odd. Therefore $n \equiv 1$ or $3 \pmod 6$ is necessary. In this lemma, we assume the vertex set of $K_{n,n}$ has partite sets $\{0_1, 1_1, \ldots, (n-1)_1\}$ and $\{0_2, 1_2, \ldots, (n-1)_2\}$.

We now consider cases. In each case, the vertex labels are reduced modulo n and the collection of hexagons form a decomposition of $K_{n,n}$.
Case 1. Suppose $n \equiv 1 \pmod{12}$, say $n = 12k+1$. Consider the hexagons: $\{[i_1, (12j+i)_2, (12k+i)_1, (12j+1+i)_2, (12k-1+i)_1, (12j+4+i)_2], [i_1, (12j+5+i)_2, (12k-1+i)_1, (12j+7+i)_2, (12k+i)_1, (12j+10+i)_2] \mid i = 0,1,\ldots,12k; j = 0,1,\ldots,k-1\}$. In this case, $E(M) = \{(i_1, (12k+i)_2) \mid i = 0,1,\ldots,12k\}$.
Case 2. Suppose $n \equiv 7 \pmod{12}$, say $n = 12k+7$. Consider the hexagons: $\{[i_1, (12j+i)_2, (12k+6+i)_1, (12j+1+i)_2, (12k+5+i)_1, (12j+4+i)_2], [i_1, (12j+5+i)_2, (12k+5+i)_1, (12j+7+i)_2, (12k+6+i)_1, (12j+10+i)_2] \mid i = 0,1,\ldots,12k+6; j = 0,1,\ldots,k-1\} \cup \{[i_1, (12k+i)_2, (12k+6+i)_1, (12k+1+i)_2, (12k+5+i)_1, (12k+4+i)_2] \mid i = 0,1,\ldots,12k+6\}$. In this case, $E(M) = \{(i_1, (12k+5+i)_2) \mid i = 0,1,\ldots,12k+6\}$.
Case 3. Suppose $n \equiv 3 \pmod{36}$, say $n = 36k+3$. Consider the hexagons: $\{[i_1, (12j+i)_2, (36k+2+i)_1, (12j+1+i)_2, (36k+1+i)_1, (12j+4+i)_2], [i_1, (12j+5+i)_2, (36k+1+i)_1, (12j+7+i)_2, (36k+2+i)_1, (12j+10+i)_2] \mid i = 0,1,\ldots,36k+2; j = 0,1,\ldots,k-1\} \cup \{[i_1, (12j+12k+7+i)_2, (36k+2+i)_1, (12j+12k+8+i)_2, (36k+1+i)_1, (12j+12k+11+i)_2], [i_1, (12j+12k+12+i)_2, (36k+1+i)_1, (12j+12k+14+i)_2, (36k+2+i)_1, (12j+12k+17+i)_2] \mid i = 0,1,\ldots,36k+2; j = 0,1,\ldots,k-2\} \cup \{[i_1, (12j+24k+14+i)_2, (36k+2+i)_1, (12j+24k+15+i)_2, (36k+1+i)_1, (12j+24k+18+i)_2], [i_1, (12j+24k+19+i)_2, (36k+1+i)_1, (12j+24k+21+i)_2, (36k+2+i)_1, (12j+24k+24+i)_2] \mid$

$i = 0, 1, \ldots, 36k+2; j = 0, 1, \ldots, k-2\} \cup \{[i_1, (12k+i)_2, (36k+1+i)_1, (12k+2+i)_2, (36k+2+i)_1, (12k+5+i)_2], [i_1, (24k-5+i)_2, (36k+2+i)_1, (24k-4+i)_2, (36k+1+i)_1, (24k-1+i)_2], [i_1, (24k+i)_2, (36k+i)_1, (24k+2+i)_2, (36k+1+i)_1, (24k+6+i)_2], [i_1, (24k+7+i)_2, (36k+1+i)_1, (24k+9+i)_2, (36k+2+i)_1, (24k+12+i)_2] \mid i = 0, 1, \ldots, 36k+2\} \cup \{[i_1, (12k+1+i)_2, (24k+2+i)_1, i_2, (12k+1+i)_1, (24k+2+i)_2] \mid i = 0, 1, \ldots, 12k\}$. In this case, $E(M) = \{(i_1, (36k+2+i)_2) \mid i = 0, 1, \ldots, 36k+2\}$.

Case 4. Suppose $n \equiv 9 \pmod{36}$, say $n = 36k+9$. Consider the hexagons: $\{[i_1, (12j+i)_2, (36k+8+i)_1, (12j+1+i)_2, (36k+7+i)_1, (12j+4+i)_2], [i_1, (12j+5+i)_2, (36k+7+i)_1, (12j+7+i)_2, (36k+8+i)_1, (12j+10+i)_2] \mid i = 0, 1, \ldots, 36k+8; j = 0, 1, \ldots, k-1\} \cup \{[i_1, (12j+12k+7+i)_2, (36k+8+i)_1, (12j+12k+8+i)_2, (36k+7+i)_1, (12j+12k+11+i)_2], [i_1, (12j+12k+12+i)_2, (36k+7+i)_1, (12j+12k+14+i)_2, (36k+8+i)_1, (12j+12k+17+i)_2] \mid i = 0, 1, \ldots, 36k+8; j = 0, 1, \ldots, k-2\} \cup \{[i_1, (12j+24k+14+i)_2, (36k+8+i)_1, (12j+24k+15+i)_2, (36k+7+i)_1, (12j+24k+18+i)_2], [i_1, (12j+24k+19+i)_2, (36k+7+i)_1, (12j+24k+21+i)_2, (36k+8+i)_1, (12j+24k+24+i)_2] \mid i = 0, 1, \ldots, 36k+8; j = 0, 1, \ldots, k-2\} \cup \{[i_1, (12k+i)_2, (36k+8+i)_1, (12k+3+i)_2, (1+i)_1, (12k+6+i)_2], [i_1, (24k-5+i)_2, (36k+8+i)_1, (24k-4+i)_2, (36k+7+i)_1, (24k-1+i)_2], [i_1, (24k+i)_2, (36k+7+i)_1, (24k+1+i)_2, (36k+6+i)_1, (24k+5+i)_2], [i_1, (24k+7+i)_2, (36k+7+i)_1, (24k+9+i)_2, (36k+8+i)_1, (24k+12+i)_2], [i_1, (36k+2+i)_2, (36k+8+i)_1, (36k+3+i)_2, (36k+7+i)_1, (36k+6+i)_2] \mid i = 0, 1, \ldots, 36k+8\} \cup \{[i_1, (12k+3+i)_2, (24k+6+i)_1, i_2, (12k+3+i)_1, (24k+6+i)_2] \mid i = 0, 1, \ldots, 12k+2\}$. In this case, $E(M) = \{(i_1, (36k+7+i)_2) \mid i = 0, 1, \ldots, 36k+8\}$.

Case 5. Suppose $n \equiv 15 \pmod{36}$, say $n = 36k+15$. Consider the hexagons: $\{[i_1, (12j+i)_2, (36k+14+i)_1, (12j+1+i)_2, (36k+13+i)_1, (12j+4+i)_2], [i_1, (12j+5+i)_2, (36k+13+i)_1, (12j+7+i)_2, (36k+14+i)_1, (12j+10+i)_2] \mid i = 0, 1, \ldots, 36k+14; j = 0, 1, \ldots, k-1\} \cup \{[i_1, (12j+12k+7+i)_2, (36k+14+i)_1, (12j+12k+8+i)_2, (36k+13+i)_1, (12j+12k+11+i)_2], [i_1, (12j+12k+12+i)_2, (36k+13+i)_1, (12j+12k+14+i)_2, (36k+14+i)_1, (12j+12k+17+i)_2] \mid i = 0, 1, \ldots, 36k+14; j = 0, 1, \ldots, k-1\} \cup \{[i_1, (12j+24k+14+i)_2, (36k+14+i)_1, (12j+24k+15+i)_2, (36k+13+i)_1, (12j+24k+18+i)_2], [i_1, (12j+24k+19+i)_2, (36k+13+i)_1, (12j+24k+21+i)_2, (36k+14+i)_1, (12j+24k+24+i)_2] \mid i = 0, 1, \ldots, 36k+14; j = 0, 1, \ldots, k-1\} \cup \{[i_1, (12k+i)_2, (36k+14+i)_1, (12k+1+i)_2, (36k+13+i)_1, (12k+4+i)_2], [i_1, (24k+7+i)_2, (36k+14+i)_1, (24k+10+i)_2, (1+i)_1, (24k+13+i)_2] \mid i = 0, 1, \ldots, 36k+14\} \cup \{[i_1, (12k+5+i)_2, (24k+10+i)_1, i_2, (12k+5+i)_1, (24k+10+i)_2] \mid i = 0, 1, \ldots, 12k+4\}$. In this case, $E(M) = \{(i_1, (36k+14+i)_2) \mid i = 0, 1, \ldots, 36k+14\}$.

Case 6. Suppose $n \equiv 21 \pmod{36}$, say $n = 36k+21$. Consider the hexagons: $\{[i_1, (12j+i)_2, (36k+20+i)_1, (12j+1+i)_2, (36k+19+i)_1, (12j+4+i)_2], [i_1, (12j+5+i)_2, (36k+19+i)_1, (12j+7+i)_2, (36k+20+i)_1, (12j+10+i)_2] \mid i = 0, 1, \ldots, 36k+20; j = 0, 1, \ldots, k-1\} \cup \{[i_1, (12j+12k+19+$

$i)_2, (36k+20+i)_1, (12j+12k+20+i)_2, (36k+19+i)_1, (12j+12k+23+$ $i)_2], [i_1, (12j+12k+24+i)_2, (36k+19+i)_1, (12j+12k+26+i)_2, (36k+$ $20+i)_1, (12j+12k+29+i)_2] \mid i=0,1,\dots,36k+20; j=0,1,\dots,k-2\} \cup$ $\{[i_1, (12j+24k+26+i)_2, (36k+20+i)_1, (12j+24k+27+i)_2, (36k+19+$ $i)_1, (12j+24k+30+i)_2], [i_1, (12j+24k+31+i)_2, (36k+19+i)_1, (12j+24k+$ $33+i)_2, (36k+20+i)_1, (12j+24k+36+i)_2] \mid i=0,1,\dots,36k+20; j=$ $0,1,\dots,k-2\} \cup \{[i_1, (12k+i)_2, (36k+20+i)_1, (12k+1+i)_2, (36k+19+$ $i)_1, (12k+4+i)_2], [i_1, (12k+5+i)_2, (36k+18+i)_1, (12k+7+i)_2, (36k+19+$ $i)_1, (12k+11+i)_2], [i_1, (12k+12+i)_2, (36k+19+i)_1, (12k+14+i)_2, (36k+$ $20+i)_1, (12k+17+i)_2], [i_1, (24k+7+i)_2, (36k+20+i)_1, (24k+8+i)_2, (36k+$ $19+i)_1, (24k+11+i)_2], [i_1, (24k+12+i)_2, (36k+18+i)_1, (24k+14+$ $i)_2, (36k+19+i)_1, (24k+18+i)_2], [i_1, (24k+19+i)_2, (36k+19+i)_1, (24k+$ $21+i)_2, (36k+20+i)_1, (24k+24+i)_2], [i_1, (36k+14+i)_2, (36k+20+i)_1, (36k+$ $15+i)_2, (36k+19+i)_1, (36k+18+i)_2] \mid i=0,1,\dots,36k+20\} \cup \{[i_1, (12k+$ $7+i)_2, (24k+14+i)_1, i_2, (12k+7+i)_1, (24k+14+i)_2] \mid i=0,1,\dots,12k+6\}$.
In this case, $E(M)=\{(i_1,(36k+19+i)_2) \mid i=0,1,\dots,36k+20\}$.

Case 7. Suppose $n \equiv 27 \pmod{36}$, say $n=36k+27$. Consider the hexagons: $\{[i_1, (12j+i)_2, (36k+26+i)_1, (12j+1+i)_2, (36k+25+i)_1, (12j+$ $4+i)_2], [i_1, (12j+5+i)_2, (36k+25+i)_1, (12j+7+i)_2, (36k+26+i)_1, (12j+$ $10+i)_2] \mid i=0,1,\dots,36k+26; j=0,1,\dots,k-1\} \cup \{[i_1, (12j+12k+19+$ $i)_2, (36k+26+i)_1, (12j+12k+20+i)_2, (36k+25+i)_1, (12j+12k+23+$ $i)_2], [i_1, (12j+12k+24+i)_2, (36k+25+i)_1, (12j+12k+26+i)_2, (36k+$ $26+i)_1, (12j+12k+29+i)_2] \mid i=0,1,\dots,36k+26; j=0,1,\dots,k-2\} \cup$ $\{[i_1, (12j+24k+26+i)_2, (36k+26+i)_1, (12j+24k+27+i)_2, (36k+25+$ $i)_1, (12j+24k+30)_2], [i_1, (12j+24k+31+i)_2, (36k+25+i)_1, (12j+24k+$ $33+i)_2, (36k+26+i)_1, (12j+24k+36+i)_2] \mid i=0,1,\dots,36k+26; j=$ $0,1,\dots,k-1\} \cup \{[i_1, (12k+i)_2, (36k+26+i)_1, (12k+1+i)_2, (36k+25+$ $i)_1, (12k+4+i)_2], [i_1, (12k+5+i)_2, (36k+25+i)_1, (12k+6+i)_2, (36k+23+$ $i)_1, (12k+11+i)_2], [i_1, (12k+12+i)_2, (36k+26+i)_1, (12k+15+i)_2, (1+$ $i)_1, (12k+18+i)_2], [i_1, (24k+7+i)_2, (36k+26+i)_1, (24k+8+i)_2, (36k+$ $25+i)_1, (24k+11+i)_2], [i_1, (24k+12+i)_2, (36k+25+i)_1, (24k+13+$ $i)_2, (36k+24+i)_1, (24k+17+i)_2], [i_1, (24k+19+i)_2, (36k+25+i)_1, (24k+$ $21+i)_2, (36k+24+i)_1, (24k+22+i)_2]\} \cup \{[i_1, (12k+9+i)_2, (24k+18+$ $i)_1, i_2, (12k+9+i)_1, (24k+18+i)_2] \mid i=0,1,\dots,12k+8\}$. In this case, $E(M)=\{(i_1,(36k+26+i)_1) \mid i=0,1,\dots,36k+26\}$.

Case 8. Suppose $n \equiv 33 \pmod{36}$, say $n=36k+33$. Consider the hexagons: $\{[i_1, (12j+i)_2, (36k+32+i)_1, (12j+1+i)_2, (36k+31+i)_1, (12j+$ $4+i)_2], [i_1, (12j+5+i)_2, (36k+31+i)_1, (12j+7+i)_2, (36k+32+i)_1, (12j+$ $10+i)_2] \mid i=0,1,\dots,36k+32; j=0,1,\dots,k-1\} \cup \{[i_1, (12j+24k+26+$ $i)_2, (36k+32+i)_1, (12j+24k+27+i)_2, (36k+31+i)_1, (12j+24k+30+$ $i)_2], [i_1, (12j+24k+31+i)_2, (36k+31+i)_1, (12j+24k+33+i)_2, (36k+$ $32+i)_1, (12j+24k+36+i)_2] \mid i=0,1,\dots,36k+32; j=0,1,\dots,k-1\} \cup$ $\{[i_1, (12k+i)_2, (36k+32+i)_1, (12k+1+i)_2, (36k+31+i)_1, (12k+4+$

$i)_2], [i_1, (12k+5+i)_2, (36k+31+i)_1, (12k+6+i)_2, (36k+30+i)_1, (12k+10+i)_2], [i_1, (12k+12+i)_2, (36k+31+i)_1, (12k+14+i)_2, (36k+32+i)_1, (12k+17+i)_2], [i_1, (36k+26+i)_2, (36k+32+i)_1, (36k+27+i)_2, (36k+31+i)_1, (36k+30+i)_2] \mid i = 0, 1, \ldots, 36k+32\} \cup \{[i_1, (12k+11+i)_2, (24k+22+i)_1, i_2, (12k+11+i)_1, (24k+22+i)_2] \mid i = 0, 1, \ldots, 12k+10\}$. In this case, $E(M) = \{(i_1, (36k+31+i)_2) \mid i = 0, 1, \ldots, 36k+32\}$. ∎

Lemma 2.2 *A maximal hexagon packing of $K_{m,n}$ where m is even and n is odd ($m \geq 4, n \geq 3$) has a leave L satisfying $|E(L)| = m + k$ where k is the smallest nonnegative integer such that $|E(K_{m,n})| - (m+k) \equiv 0 \pmod 6$.*

Proof. Since each vertex of V_m is of odd degree in $K_{m,n}$, in the leave of a packing each of these vertices will be of odd degree. Therefore in a packing of $K_{m,n}$ with leave L, it is necessary that $|E(L)| \geq m$. Since $K_{m,n}$ is a union of L and a collection of hexagons, then $|E(K_{m,n})| \equiv |E(L)| \pmod 6$. So in a maximal packing, it is necessary that $|E(L)| = m + k$ where k is as described. We now establish sufficiency.

Case 1. Suppose $m \equiv 0 \pmod 6$ and $n \equiv 1 \pmod 2$. Now $K_{m,n} = K_{m,n-3} \cup \frac{m}{3} \times (K_{3,3} \setminus M) \cup \frac{m}{3} \times M$ where the partite sets of $K_{m,n-3}$ are V_m and $V_n \setminus \{1_2, 2_2, 3_2\}$, the partite sets of the ith $K_{3,3} \setminus M$ are $\{(3i-2)_1, (3i-1)_1, (3i)_1\}$ and $\{1_2, 2_2, 3_2\}$, and M is a perfect matching of $K_{3,3}$. Now $K_{m,n-3}$ can be decomposed into hexagons by Theorem 1.1 and each $K_{3,3} \setminus M$ can be decomposed into hexagons by Lemma 2.1. Therefore a maximal packing of $K_{m,n}$ will have a leave L where $|E(L)| = m$.

Case 2. Suppose $m \equiv 0 \pmod 2$, $m \geq 4$, and $n \equiv 1 \pmod 6$. Now $K_{m,n} = K_{m,n-1} \cup S_m$ where the partite sets of $K_{m,n-1}$ are V_m and $V_n \setminus \{1_2\}$, and the edge set of S_m is $\{(i_1, 1_2) \mid i = 1, 2, \ldots, m\}$. Now $K_{m,n-1}$ can be decomposed into hexagons by Theorem 1.1. Therefore a maximal packing of $K_{m,n}$ will have a leave L where $|E(L)| = m$.

Case 3. Suppose $m \equiv 2 \pmod 6$, $m \geq 4$, and $n \equiv 3 \pmod 6$. Now $|E(K_{m,n}| - m \equiv 4 \pmod 6$, so it is necessary that a packing have a leave L with $|E(L)| \geq m + 4$. Now $K_{m,n} = K_{m,n-3} \cup (K_{m-8,3} \setminus M) \cup 2 \times C_6 \cup M \cup 12 \times K_2$ where the partite sets of $K_{m,n-3}$ are V_m and $V_n \setminus \{1_2, 2_2, 3_2\}$, the partite sets of $K_{m-8,3} \setminus M$ are $\{9_1, 10_1, \ldots, m_1\}$ and $\{1_2, 2_2, 3_2\}$, M is a collection of $m-8$ edges of $K_{m-8,3}$ as described in Case 1, $2 \times C_6 = \{[3_1, 2_2, 4_1, 3_2, 5_1, 1_2], [6_1, 3_2, 8_1, 2_2, 7_1, 1_2]\}$, and $12 \times K_2 = \{(1_1, 1_2), (1_1, 2_2), (1_1, 3_2), (2_1, 1_2), (2_1, 2_2), (2_1, 3_2), (3_1, 3_2), (4_1, 1_2), (5_1, 2_2), (6_1, 2_2), (7_1, 3_2), (8_1, 1_2)\}$. Now $K_{m,n-3}$ can be decomposed into hexagons by Theorem 1.1 and $K_{m-8,3} \setminus M$ can be decomposed into hexagons by Case 1. Therefore a maximal packing of $K_{m,n}$ will have a leave L where $|E(L)| = |E(M)| + |E(12 \times K_2)| = m + 4$.

Case 4. Suppose $m \equiv 2 \pmod 6$, $m \geq 8$, and $n \equiv 5 \pmod 6$. Now

$|E(K_{m,n})| - m \equiv 2 \pmod 6$, so it is necessary that a packing have a leave L with $|E(L)| \geq m+2$. Now $K_{m,n} = K_{m,n-5} \cup (K_{m-8,5} \setminus M) \cup 5 \times C_6 \cup 10 \times K_2$ where the partite sets of $K_{m,n-5}$ are V_m and $V_n \setminus \{1_2, 2_2, 3_2, 4_2, 5_2\}$, the partite sets of $K_{m-8,5} \setminus M$ are $\{9_1, 10_1, \ldots, m_1\}$ and $\{1_2, 2_2, 3_2, 4_2, 5_2\}$, M is a collection of $m-8$ edges as described in Case 1, $5 \times C_6 = \{[1_1, 1_2, 2_1, 2_2, 3_1, 3_2], [2_1, 3_2, 4_1, 5_2, 3_1, 4_2], [5_1, 1_2, 6_1, 2_2, 7_1, 3_2], [6_1, 3_2, 8_1, 5_2, 7_1, 4_2], [8_1, 1_2, 4_1, 2_2, 1_1, 4_2]\}$, and $10 \times K_2 = \{(1_1, 5_2), (2_1, 5_2), (3_1, 1_2), (4_1, 4_2), (5_1, 2_2), (5_1, 4_2), (5_1, 5_2), (6_1, 5_2), (7_1, 1_2), (8_1, 2_2)\}$. Now $K_{m,n-5}$ can be decomposed into hexagons by Theorem 1.1 and $K_{m-8,5} \setminus M$ can be decomposed into hexagons by Case 1. Therefore a maximal packing of $K_{m,n}$ will have a leave L where $|E(L)| = |E(M)| + |E(10 \times K_2)| = m+2$.

Case 5. Suppose $m \equiv 4 \pmod 6$ and $n \equiv 3 \pmod 6$. As in Case 4, a packing with leave L satisfies $|E(L)| \geq m+2$. Now $K_{m,n} = K_{m,n-3} \cup (K_{m-4,3} \setminus M) \cup C_6 \cup M \cup 6 \times K_2$ where the partite sets of $K_{m,n-3}$ are V_m and $V_n \setminus \{1_2, 2_2, 3_2\}$, the partite sets of $K_{m-4,3} \setminus M$ are $V_m \setminus \{1_1, 2_1, 3_1, 4_1\}$ and $\{1_2, 2_2, 3_2\}$, M is a collection of $m-4$ edges of $K_{m-4,3}$ as described in Case 1, $C_6 = [1_1, 1_2, 2_1, 2_2, 3_1, 3_2]$, and $6 \times K_2 = \{(1_1, 2_2), (2_1, 3_2), (3_1, 1_2), (4_1, 1_2), (4_1, 2_2), (4_1, 3_2)\}$. Now $K_{m,n-3}$ can be decomposed into hexagons by Theorem 1.1 and $K_{m-4,3} \setminus M$ can be decomposed into hexagons by Case 1. Therefore a maximal packing of $K_{m,n}$ will have a leave L were $|E(L)| = |E(M)| + |E(6 \times K_2)| = m+2$.

Case 6. Suppose $m \equiv 4 \pmod 6$ and $n \equiv 5 \pmod 6$. As in Case 3, a packing with leave L satisfies $|E(L)| \geq m+4$. Now $K_{m,n} = K_{m,n-5} \cup (K_{m-4,5} \setminus M) \cup 2 \times C_6 \cup M \cup 8 \times K_2$ where the partite sets of $K_{m,n-5}$ are V_m and $V_n \setminus \{1_2, 2_2, 3_2, 4_2, 5_2\}$, the partite sets of $K_{m-4,5} \setminus M$ are $V_m \setminus \{1_1, 2_1, 3_1, 4_1\}$ and $\{1_2, 2_2, 3_2, 4_2, 5_2\}$, M is a collection of $m-4$ edges of $K_{m-4,5}$ as described in Case 1, $2 \times C_6 = \{[1_1, 1_2, 2_1, 2_2, 3_1, 3_2], [2_1, 3_2, 4_1, 5_2, 3_1, 4_2]\}$ and $8 \times K_2 = \{(1_1, 2_2), (1_1, 4_2), (1_1, 5_2), (2_1, 5_2), (3_1, 1_2), (4_1, 1_2), (4_1, 2_2), (4_1, 4_2)\}$. Now $K_{m,n-5}$ can be decomposed into hexagons by Theorem 1.1 and $K_{m-4,5} \setminus M$ can be decomposed into hexagons by Case 1. Therefore a maximal packing of $K_{m,n}$ will have a leave L where $|E(L)| = |E(M)| + |E(8 \times K_2)| = m+4$. ∎

Lemma 2.3 *A maximal hexagon packing of $K_{m,n}$ where m and n are both odd, $m \geq n \geq 3$, has a leave L satisfying $|E(L)| = m+k$ where k is the smallest nonnegative integer such that $|E(K_{m,n})| - (m+k) \equiv 0 \pmod 6$.*

Proof. The necessary conditions follow as in Lemma 2.2. We now establish sufficiency.

Case 1. Suppose $m \equiv 1 \pmod 6$, $n \equiv 1 \pmod 6$, and $m \geq n$. Now $K_{m,n} = (K_{n,n} \setminus M_1) \cup (K_{m-n,n} \setminus M_2) \cup M_1 \cup M_2$ where the partite sets of $K_{n,n} \setminus M_1$ are $\{1_1, 2_1, \ldots, n_1\}$ and V_n, M_1 is a perfect matching of $K_{n,n}$, the partite sets of $K_{m-n,n} \setminus M_2$ are $\{(n+1)_1, (n+2)_1, \ldots, m_1\}$ and V_n, and M_2 is a set of $m-n$ edges of $K_{m-n,n}$ as described in Case 1 of Lemma 2.2. Now $K_{n,n} \setminus M_1$ can be decomposed into hexagons by Lemma 2.1 and $K_{m-n,n} \setminus M_2$ can be decomposed into hexagons by Lemma 2.2 Case 1. Therefore a maximal packing of $K_{m,n}$ will have a leave L where $|E(L)| = m$.

Case 2. Suppose $m \equiv 1 \pmod 6$, $n \equiv 3 \pmod 6$, and $m > n$. As in Case 4 of Lemma 2.2, a packing with leave L satisfies $|E(L)| \geq m+2$. Now $K_{m,n} = (K_{n,n} \setminus M_1) \cup (K_{m-n,n} \setminus M_2) \cup M_1 \cup M_2$ where the partite sets of $K_{n,n} \setminus M_1$ are $\{1_1, 2_1, \ldots, n_1\}$ and V_n, M_1 is a perfect matching of $K_{n,n}$, the partite sets of $K_{m-n,n} \setminus M_2$ are $\{(n+1)_1, (n+2)_1, \ldots, m_1\}$ and V_n, and M_2 is a set of $m-n+2$ edges of $K_{m-n,n}$ as described in Case 5 of Lemma 2.2. Now $K_{n,n} \setminus M_1$ can be decomposed into hexagons by Lemma 2.1 and $K_{m-n,n} \setminus M_2$ can be decomposed into hexagons by Lemma 2.2 Case 5. Therefore a maximal packing of $K_{m,n}$ will have a leave L where $|E(L)| = m+2$.

Case 3. Suppose $m \equiv 1 \pmod 6$, $n \equiv 5 \pmod 6$, and $m > n$. As in Case 3 of Lemma 2.2, a packing with leave L satisfies $|E(L)| \geq m+4$. Now $K_{m,n} = (K_{n-4,n-4} \setminus M_1) \cup (K_{m-n+4,n} \setminus M_2) \cup (K_{m,4} \setminus M_3) \cup M_1 \cup M_2 \cup M_3$ where the partite sets of $K_{n-4,n-4} \setminus M_1$ are $\{1_1, 2_1, \ldots, (n-4)_1\}$ and $V_n \setminus \{(n-3)_2, (n-2)_2, (n-1)_2, n_2\}$, M_1 is a perfect matching of $K_{n-4,n-4}$, the partite sets of $K_{m-n+4,n} \setminus M_2$ are $\{(n-3)_1, (n-2)_1, \ldots, m_1\}$ and V_n, and M_2 is a set of $m-n+4$ edges of $K_{m-n+4,n}$ as described in Case 1 of Lemma 2.2, the partite sets of $K_{m,4} \setminus M_3$ are V_m and $\{(n-3)_2, (n-2)_2, (n-1)_2, n_2\}$, and M_3 is a set of 4 edges as described in Case 1 of Lemma 2.2. Now $K_{n-4,n-4} \setminus M_1$ can be decomposed into hexagons by Lemma 2.1, and $K_{m-n+4,n} \setminus M_2$ and $K_{m,4} \setminus M_3$ can be decomposed into hexagons by Lemma 2.2 Case 1. Therefore a maximal packing of $K_{m,n}$ will have a leave L where $|E(L)| = m+4$.

Case 4. Suppose $m \equiv 3 \pmod 6$, $n \equiv 1 \pmod 6$, and $m > n$. This case follows the same as Case 1, but in this case $K_{m-n,n} \setminus M_2$ can be decomposed into hexagons by Lemma 2.2 Case 2. Again, the leave L in a maximal packing satisfies $|E(L)| = m$.

Case 5. Suppose $m \equiv 3 \pmod 6$, $n \equiv 3 \pmod 6$, and $m \geq n$. This case follows the same as Case 1. Again, the leave L in a maximal packing satisfies $|E(L)| = m$.

Case 6. Suppose $m \equiv 3 \pmod 6$, $n \equiv 5 \pmod 6$, and $m > n$. Now $K_{m,n} = K_{n-5,4} \cup K_{m-n+4,n-5} \cup K_{m-n-4,4} \cup (K_{n-4,n-4} \setminus M) \cup 6 \times C_6 \cup 8 \times K_2 \cup S_{m-n-4}$ where the partite sets of $K_{n-5,4}$ are $\{1_1, 2_1, \ldots, (n-5)_1\}$ and $\{(m-3)_2, (m-2)_2, (m-1)_2, m_2\}$, the partite sets of $K_{m-n+4,n-5}$ are $\{(n-3)_1, (n-2)_1, \ldots, m_1\}$ and $V_n \setminus \{(n-4)_2, (n-3)_2, (n-2)_2, (n-1)_2, n_2\}$, the partite sets of $K_{m-n-4,4}$ are $\{(n+5)_1, (n+6)_1, \ldots, m_1\}$ and $\{(n-3)_2, (n-2)_2, (n-1)_2, n_2\}$, the partite sets of $K_{n-4,n-4} \setminus M$ are $\{1_1, 2_1, \ldots, (n-4)_1\}$ and $V_n \setminus \{(n-3)_2, (n-2)_2, (n-1)_2, n_2\}$, M is a perfect matching of $K_{n-4,n-4}$ containing the edge $((n-4)_1, (n-4)_2)$, $6 \times$

$C_6 = \{[(n-3)_1, (n-3)_2, (n-4)_1, (n-1)_2, n_1, (n-2)_2], [(n-1)_1, (n-3)_2, n_1, (n-4)_2, (n-2)_1, n_2], [(n-3)_1, (n-1)_2, (n-1)_1, (n-2)_2, (n-4)_1, n_2], [(n+1)_1, (n-3)_2, (n+2)_1, (n-4)_2, (n+3)_1, (n-2)_2], [(n+2)_1, (n-2)_2, (n+4)_1, n_2, (n+3)_1, (n-1)_2], [(n-2)_1, (n-3)_2, (n+4)_1, (n-4)_2, (n+1)_1, (n-1)_2]\}$, $8 \times K_2 = \{((n-3)_1, (n-4)_2), ((n-2)_1, (n-2)_2), ((n-1)_1, (n-4)_2), (n_1, n_2), ((n+1)_1, n_2), ((n+2)_1, n_2), ((n+3)_1, (n-3)_2), ((n+4)_1, (n-1)_2)\}$, and $E(S_{m-n-4}) = \{((n-4)_2, (n+4+i)_1) \mid i = 1, 2, \ldots, m-n-4\}$. Now $K_{n-5,4}$, $K_{m-n+4,n-5}$, and $K_{m-n-4,4}$ can be decomposed into hexagons by Theorem 1.1, and $K_{n-4,n-4} \setminus M$ can be decomposed into hexagons by Lemma 2.1. Therefore a maximal packing of $K_{m,n}$ will have a leave L where $|E(L)| = |E(M)| + |E(8 \times K_2)| + |E(S_{m-n-4})| = m$.

Case 7. Suppose $m \equiv 5 \pmod 6$, $n \equiv 1 \pmod 6$, and $m > n$. This case follows the same as Case 1, but in this case $K_{m-n,n} \setminus M_2$ can be decomposed into hexagons by Lemma 2.2 Case 2. Again, the leave L in a maximal packing satisfies $|E(L)| = m$.

Case 8. Suppose $m \equiv 5 \pmod 6$, $n \equiv 3 \pmod 6$, and $m \geq n$. As in Case 3 of Lemma 2.2, a packing with leave L satisfies $|E(L)| \geq m + 4$. Now $K_{m,n} = (K_{n,n} \setminus M_1) \cup (K_{m-n,n} \setminus M_2) \cup M_1 \cup M_2$ where the partite sets of $K_{n,n} \setminus M_1$ are $\{1_1, 2_1, \ldots, n_1\}$ and V_n, M_1 is a perfect matching of $K_{n,n}$, the partite sets of $K_{m-n,n} \setminus M_2$ are $\{(n+1)_1, (n+2)_1, \ldots, m_1\}$ and V_n, and M_2 is a set of $m - n + 4$ edges of $K_{m-n,n}$ as described in Case 3 of Lemma 2.2. Now $K_{n,n} \setminus M_1$ can be decomposed into hexagons by Lemma 2.1 and $K_{m-n,n} \setminus M_2$ can be decomposed into hexagons by Lemma 2.2 Case 3. Therefore a maximal packing of $K_{m,n}$ will have a leave L where $|E(L)| = |E(M_1)| + |E(M_2)| = m + 4$.

Case 9. Suppose $m \equiv 5 \pmod 6$, $n \equiv 5 \pmod 6$, and $m \geq n$. As in Case 4 of Lemma 2.2, a packing with leave L satisfies $|E(L)| \geq m+2$. Now $K_{m,n} = K_{4,n-5} \cup K_{n-5,4} \cup (K_{n-4,n-4} \setminus M_1) \cup (K_{m-n,n} \setminus M_2) \cup 3 \times C_6 \cup 6 \times K_2 \cup M_1 \cup M_2$ where the partite sets of $K_{4,n-5}$ are $\{1_1, 2_1, 3_1, 4_1\}$ and $V_n \setminus \{1_2, 2_2, 3_2, 4_2, 5_2\}$, the partite sets of $K_{n-5,4}$ are $V_m \setminus \{1_1, 2_1, 3_1, 4_1, 5_1\}$ and $\{1_2, 2_2, 3_2, 4_2\}$, the partite sets of $K_{n-4,n-4} \setminus M_1$ are $\{5_1, 6_1, \ldots, n_1\}$ and $V_n \setminus \{1_2, 2_2, 3_2, 4_2\}$, M_1 is a perfect matching of $K_{n-4,n-4}$ which contains the edge $(5_1, 5_2)$, the partite sets of $K_{m-n,n} \setminus M_2$ are $\{(n+1)_1, (n+2)_1, \ldots, m_1\}$ and V_n, M_2 is a set of $m-n$ edges as described in Case 1 of Lemma 2.2, $3 \times C_6 = \{[1_1, 2_2, 2_1, 3_2, 5_1, 4_2], [3_1, 2_2, 5_1, 1_2, 4_1, 5_2], [1_1, 3_2, 3_1, 4_2, 2_1, 5_2]\}$, and $6 \times = \{(1_1, 1_2), (2_1, 1_2), (3_1, 1_2), (4_1, 2_2), (4_1, 3_2), (4_1, 4_2)\}$. Now $K_{4,n-5}$ and $K_{n-5,4}$ can be decomposed into hexagons by Theorem 1.1, $K_{n-4,n-4} \setminus M_1$ can be decomposed into hexagons by Lemma 2.1, and $K_{m-n,n} \setminus M_2$ can be decomposed into hexagons by Lemma 2.2 Case 1. Therefore a maximal packing of $K_{m,n}$ will have a leave L where $|E(L)| = |E(M_1)| + |E(M_2)| + |E(6 \times K_2)| = m + 2$. ∎

Lemma 2.4 *A maximal hexagon packing of $K_{m,n}$ where m and n are even, $m, n \geq 4$, has a leave L satisfying:*
(1) $|E(L)| = 0$ when $m \equiv 0 \pmod 6$,
(2) $|E(L)| = 4$ when $m \equiv n \equiv 2 \pmod 6$ or $m \equiv n \equiv 4 \pmod 6$, and
(3) $|E(L)| = 8$ when $m \equiv 2 \pmod 6$ and $n \equiv 4 \pmod 6$.

Proof. We consider cases.
Case 1. Suppose $m \equiv 0 \pmod 6$, $n \equiv 0 \pmod 2$, and $n \geq 4$. Then $K_{m,n}$ can be decomposed into hexagons by Theorem 1.1 and in a maximal packing, $|E(L)| = 0$.
Case 2. Suppose $m \equiv n \equiv 2 \pmod 6$, $m, n \geq 4$, Now $|E(K_{m,n})| \equiv 4 \pmod 6$, so it is necessary that a packing have leave L with $|E(L)| \geq 4$. Now $K_{m,n} = K_{m-8,n} \cup K_{8,n-8} \cup 10 \times C_6 \cup C_4$ where the partite sets of $K_{m-8,n}$ are $\{9_1, 10_1, \ldots, m_1\}$ and V_n, the partite sets of $K_{8,n-8}$ are $\{1_1, 2_1, \ldots, 8_1\}$ and $\{9_2, 10_2, \ldots, n_2\}$, $10 \times C_6 = \{[2_1, 2_2, 4_1, 8_2, 1_1, 3_2], [3_1, 3_2, 5_1, 1_2, 2_1, 4_2], [4_1, 4_2, 6_1, 2_2, 3_1, 5_2], [5_1, 5_2, 7_1, 3_2, 4_1, 6_2], [7_1, 7_2, 1_1, 5_2, 6_1, 8_2], [8_1, 8_2, 2_1, 6_2, 7_1, 1_2], [8_1, 3_2, 6_1, 1_2, 4_1, 7_2], [6_1, 6_2, 1_1, 4_2, 5_1, 7_2], [1_1, 1_2, 3_1, 8_2, 5_1, 3_2], [2_1, 5_2, 8_1, 6_2, 3_1, 7_2]\}$, and $C_4 = [7_1, 2_2, 8_1, 4_2]$. Now $K_{m-8,n}$ and $K_{8,n-8}$ can be decomposed into hexagons by Theorem 1.1. Therefore a maximal packing of $K_{m,n}$ will have a leave L where $|E(L)| = |E(C_4)| = 4$.
Case 3. Suppose $m \equiv n \equiv 4 \pmod 6$, As in Case 2, a packing with leave L satisfies $|E(L)| \geq 4$. Now $K_{m,n} = K_{m-4,n} \cup K_{4,n-4} \cup 2 \times C_6 \cup C_4$ where the partite sets of $K_{m-4,n}$ are $V_m \setminus \{1_1, 2_1, 3_1, 4_1\}$ and V_n, the partite sets of $K_{4,n-4}$ are $\{1_1, 2_1, 3_1, 4_1\}$ and $V_n \setminus \{1_2, 2_2, 3_2, 4_2\}$, $2 \times C_6 = \{[1_1, 1_2, 2_1, 2_2, 3_1, 3_2], [4_1, 1_2, 3_1, 4_2, 2_1, 3_2]$, and $C_4 = [1_1, 2_2, 4_1, 4_2]$. Now $K_{m-4,n}$ and $K_{4,n-4}$ can be decomposed into hexagons by Theorem 1.1. Therefore a maximal packing of $K_{m,n}$ will have a leave L where $|E(L)| \equiv |E(C_4)| = 4$.
Case 4. Suppose $m \equiv 2 \pmod 6$, $m \geq 8$, and $n \equiv 4 \pmod 6$. Now $|E(K_{m,n})| \equiv 2 \pmod 6$, so it is necessary that a packing have a leave L with $|E(L)| \geq 2$. Now each vertex of $K_{m,n}$ is of even degree and each vertex of C_6 is of even degree, so for any packing, the vertices of L must be of even degree. However, we cannot have this property when $|E(L)| = 2$. So it is necessary that a packing have a leave L with $|E(L)| \geq 8$. Now $K_{m,n} = K_{8,n-4} \cup K_{m-8,n} \cup 4 \times C_6 \cup 2 \times C_4$ where the partite sets of $K_{8,n-4}$ are $\{1_1, 2_1, \ldots, 8_1\}$ and $V_n \setminus \{1_2, 2_2, 3_2, 4_2\}$, the partite sets of $K_{m-8,n}$ are $\{9_1, 10_1, \ldots, m_1\}$ and V_n, $4 \times C_6 = \{[1_1, 1_2, 2_1, 2_2, 3_1, 3_2], [4_1, 1_2, 3_1, 4_2, 2_1, 3_2], [5_1, 1_2, 6_1, 2_2, 7_1, 3_2], [8_1, 1_2, 7_1, 4_2, 6_1, 3_2]\}$, and $2 \times C_4 = \{[1_1, 2_2, 4_1, 4_2], [2_2, 5_1, 4_2, 8_1]\}$. Now $K_{8,n-4}$ and $K_{m-8,n}$ can be decomposed into hexagons by Theorem 1.1. Therefore a maximal packing of $K_{m,n}$ will have a leave L where $|E(L)| = |E(2 \times C_4)| = 8$. ∎

Lemmas 2.2–2.4 combine to give our main result.

Theorem 2.1 *A maximal hexagon packing of $K_{m,n}$ with leave L satisfies*
(1) when $m \equiv 0$ (mod 2) and $n \equiv 1$ (mod 2), $|E(L)| = m+k$ where k is the smallest nonnegative integer such that $|E(K_{m,n})| - (m+k) \equiv 0$ (mod 6),
(2) when $m \equiv n \equiv 1$ (mod 2)and $m \geq n$, $|E(L)| = m+k$ where k is the smallest nonnegative integer such that $|E(K_{m,n})| - (m+k) \equiv 0$ (mod 6),
(3) when $m \equiv 0$ (mod 6) and $n \equiv 0$ (mod 2), $|E(L)| = 0$,
(4) when $m \equiv n \equiv 2$ (mod 6) or $m \equiv n \equiv 4$ (mod 6), then $|E(L)| = 4$, and
(5) when $m \equiv 2$ (mod 6) and $n \equiv 4$ (mod 6), then $|E(L)| = 8$.

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