Decompositions of the Complete Symmetric Digraph into Orientations of the 4-Cycle with a Pendant Edge

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Abstract. There are twenty orientations of the 4-cycle with a pendant edge. We give necessary and sufficient conditions for the decomposition of the complete symmetric digraph on \( v \) vertices into each of these digraphs.

1 Introduction

A \( g \)-decomposition of (simple) graph \( G \) is a set \( \gamma = \{g_1, g_2, \ldots, g_n\} \) of isomorphic copies of graph \( g \), called blocks, such that \( V(g_i) \subset V(G) \) for \( i = 1, 2, \ldots, n \), \( E(g_i) \cap E(g_j) = \emptyset \) for \( i \neq j \), and \( \bigcup_{i=1}^{n} g_i = G \). That is, a \( g \)-decomposition of \( G \) is a partitioning of \( E(G) \) into the edge sets \( E(g_1), E(g_2), \ldots, E(g_n) \). The definition of a decomposition of a digraph is similarly defined, with edge sets replaced with arc sets.

Graph and digraph decompositions are a widely studied area of design theory [3]. Probably the best known graph decompositions involve decompositions of the complete graph on \( v \) vertices, \( K_v \), into cycles of a given length. For example, a 3-cycle decomposition of \( K_v \) is equivalent to a Steiner triple system and exists if and only if \( v \equiv 1 \) or \( 3 \pmod{6} \) [9]. It is well known that a 4-cycle decomposition of \( K_v \) exists if and only if \( v \equiv 1 \pmod{8} \). Many other decompositions of \( K_v \) into copies of small graphs have been studied [4]. Of particular interest to us, is a decomposition of \( K_v \) into copies of \( L = C_3 \cup \{e\} \), the 3-cycle with a pendant edge. These exist if and only if \( v \equiv 0 \) or \( 1 \pmod{8} \) [2]. A decomposition of \( K_v \) into copies of \( H = C_4 \cup \{e\} \), the 4-cycle with a pendant edge, exists if and only if \( v \equiv 0 \) or \( 1 \pmod{5} \), \( v \neq 10 \) [1].

Two nonisomorphic digraphs are determined by putting an orientation on a 3-cycle: The 3-circuit and the transitive triple. A decomposition of the complete symmetric digraph on \( v \) vertices, \( D_v \), into 3-circuits is equivalent to a Mendelsohn triple system of order \( v \), and such systems exist if and only if \( v \equiv 0 \) or \( 1 \pmod{3} \), \( v \neq 6 \) [8]. A decomposition of \( D_v \) into transitive triples is equivalent to a directed triple system of order \( v \) and such a system exists if and only if \( v \equiv 0 \) or \( 1 \pmod{3} \) [7]. There are four orientations of a 4-cycle. These are given in Figure 1. A 4-circuit decomposition of \( D_v \) exists if and only if \( v \equiv 0 \) or \( 1 \pmod{4} \), \( v \neq 4 \) [10]. An \( X \)-decomposition of \( D_v \) exists if and only if \( v \equiv 0 \) or \( 1 \pmod{4} \), \( v \neq 5 \), a \( Y \)-decomposition of \( D_v \)
exists if and only if $v \equiv 0$ or 1 (mod 4), $v \notin \{4, 5\}$, and a $Z$-decomposition of $D_v$ exists if and only if $v \equiv 1$ (mod 4) [6].

Eight digraphs are determined by putting an orientation on the graph $L = C_3 \cup \{e\}$. Necessary and sufficient conditions for the existence of a decomposition of $D_v$ into each of the orientations of $L = C_3 \cup \{e\}$ are given in [5]. Twenty digraphs are determined by putting an orientation on the graph $H = C_4 \cup \{e\}$. Half of these digraphs are given in Figure 2 (the others are the converses of those given). We denote the digraph of Figure 2 which is labeled $C_4$; as $[a, b, c, d; e]_{C_4}$, the digraph which is labeled $X_{14}$ as $[a, b, c, d; e]_{X_{14}}$, and so forth. The purpose of this paper is to give necessary and sufficient conditions for the existence of a decomposition of $D_v$ into each of the orientations of $H$.

**Figure 1.** The four orientations of a 4-cycle are the 4-circuit and the graphs $X$, $Y$, and $Z$ given here.

2 The Decompositions

In this section, we give necessary and sufficient conditions for the existence of a decomposition of the complete digraph into each of the twenty digraphs of Figure 2. Decompositions into the converse of these digraphs follows trivially. Since $D_v$ has $(v(v - 1))$ arcs and each digraph of Figure 2 has 5 arcs, we have the following necessary condition.

**Lemma 2.1** If a decomposition of $D_v$ into one of the digraphs of Figure 2 exists, then $v \equiv 0$ or 1 (mod 5).

We now show that certain decompositions do not exist for some small values of $v$.

**Lemma 2.2** The following decompositions of $D_v$ do not exist: an $X_{2i}$ decomposition of $D_5$, an $X_{2e}$ decomposition of $D_5$, a $Y_{1i}$ decomposition of $D_v$ for $v = 5, 6$, a $Y_{1e}$ decomposition of $D_v$ for $v = 5, 6$, a $Z_{1i}$ decomposition of $D_5$, a $Z_{1e}$ decomposition of $D_5$, a $Z_{2i}$ decomposition of $D_5$, and a $Z_{2e}$ decomposition of $D_5$. 

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Figure 2. Here are half of the orientations of a 4-cycle with a pendant edge. The converses of these are the other ten orientations. We denote the converse of each $A_{ni}$ as $A_{ne}$ for $A \in \{C, X, Y, Z\}$ and $n \in \{1, 2, 3, 4\}$.
Proof. Since a decomposition of $D_v$ into a particular given digraph is equivalent to a decomposition of $D_v$ into the converse of the given digraph, we only establish the nonexistence of decompositions in the cases for which the pendant arc points into the oriented $4$-cycle.

Case 1. An $X_{2i}$ decomposition of $D_5$ does not exist. A computational argument for this result is given in the appendix.

Case 2. A $Y_{14}$ decomposition of $D_5$ does not exist. Suppose, to the contrary, that such a decomposition does in fact exist. We claim that such a decomposition cannot have two blocks of the form $B_1 = [a_1, b_1, x, d_1; e_1]_{Y_{14}}$ and $B_2 = [a_2, b_2, x, d_2; e_2]_{Y_{14}}$. For, if the decomposition does contain $B_1$ and $B_2$ where $B_1 \neq B_2$, then vertex $x$ has out-degree $4$ in digraph $B_1 \cup B_2$. The decomposition of $D_5$ consists of $4$ copies of $Y_{14}$. The remaining two blocks containing $x$ must be of the form $B_3 = [x, b_3, c_3, d_3; e_3]_{Y_{14}}$ and $B_4 = [x, b_4, c_4, d_4; e_4]_{Y_{14}}$. Now $x$ has out-degree $4$ and in-degree $6$ in digraph $B_1 \cup B_2 \cup B_3 \cup B_4$. This is a contradiction because in $D_5$, $x$ has in-degree $4$ and therefore the decomposition cannot include blocks of the forms $B_1$ and $B_2$.

Let $V(D_5) = \{1, 2, 3, 4, 5\}$. Without loss of generality, one block of the decomposition is $B_1 = [1, 2, 3, 4, 5]_{Y_{14}}$. In $B_1$, vertex $1$ has in-degree $3$ and out-degree $0$. Next, in order to get the in-degree of vertex $1$ up to $4$, vertex $1$ must be in a block $B_2$ either of the form $[a, 1, c, d; e]_{Y_{14}}$ or of the form $[a, b, c, 1; c]_{Y_{14}}$. But then $c \neq 1$ (1 is already a vertex of block $B_2$), $c \neq 2$ (since arc $[2, 1]$ is in $B_1$), $c \neq 3$ (by the earlier claim applied to blocks $B_1$ and $B_2$), $c \neq 4$ (since arc $[4, 1]$ is in $B_1$), and $c \neq 5$ (since arc $[5, 1]$ is in $B_1$). These contradictions imply that no such decomposition exists.

Case 3. A $Y_{14}$ decomposition of $D_6$ does not exist. Suppose, to the contrary, that such a decomposition does in fact exist. Such a decomposition cannot have two blocks of the form $B_1 = [a_1, b_1, x, d_1; e_1]_{Y_{14}}$ and $B_2 = [a_2, b_2, x, d_2; e_2]_{Y_{14}}$. Suppose it does contain blocks $B_1$ and $B_2$, where $B_1 \neq B_2$. (1) Suppose $B_3 = [a_3, x, c_3, d_3; e_3]_{Y_{14}}$ is a block. Then $x$ has out-degree $5$ and in-degree $1$ in $B_1 \cup B_2 \cup B_3$. Then $x$ is in $2$ more blocks. The only way to get the in-degree of $x$ up to $5$ is to have $x$ in a block of the form $B_3 = [x, b_4, c_4, d_4; e_4]_{Y_{14}}$. Now $x$ has out-degree $5$, in-degree $4$ in $B_1 \cup B_2 \cup B_3 \cup B_4$. But $x$ cannot be in $B_5$ such that $x$ has out-degree $0$ and in-degree $1$ in $B_5$. So no such decomposition exists. (2) Suppose $B_3 = [a_3, b_3, c_3; x; e_3]_{Y_{14}}$. This leads to the same contradiction as (1). (3) So the remaining $3$ blocks containing $x$ must be of the forms: $[x, b, c, d; e]_{Y_{14}}$, $[a, b, x, d; e]_{Y_{14}}$, or $[a, b, c, d; x]_{Y_{14}}$. But no combination of these three blocks with $B_1$ and
$B_2$ yield out-degree 5, in-degree 5 for $x$. Hence no such decomposition exists.

Case 4. A $Z_{14}$ decomposition of $D_5$ does not exist. Suppose, to the contrary, that such a decomposition does in fact exist. Let $[a_i, b_i, c_i, d_i; e_i]_{Z_{14}}$ be a block of the decomposition. Then vertex $a_i$ is of in-degree 3 in this block and of in-degree 4 in $D_5$. Since no vertex of $Z_{14}$ is of in-degree 1, then no such decomposition exists.

Case 5. A $Z_{2i}$ decomposition of $D_5$ does not exist. Suppose, to the contrary, that such a decomposition does in fact exist. Then there are 4 copies of $Z_{2i}$ in this decomposition. In $Z_{2i} = [a, b, c, d; e]_{Z_{2i}}$, only one vertex (vertex $e$) is of odd out-degree. So if $B_i = [a_i, b_i, c_i, d_i; e_i]_{Z_{2i}}$ is a block of such a decomposition then, since vertex $x$ is of out-degree 4 in $D_5$, then there must be another block of $B_2 = [1_2, b_2, c_2, d_2; x]_{Z_{2i}} \neq B_1$ in the decomposition. But then vertex $x$ has in-degree 0 and out-degree 2 in digraph $B_1 \cup B_2$. Therefore, vertex $x$ must be of in-degree 4 and out-degree 2 in the remaining two blocks. However, it is not possible for two copies of $Z_{2i}$ to satisfy this condition. Hence, no such decomposition exists.

We now show that the necessary conditions of Lemmas 2.1 and 2.2 are sufficient.

Theorem 2.3 A decomposition of $D_v$ into each of the digraphs of Figure 2 exists if and only if $v \equiv 0$ or 1 (mod 5), with the following exceptions: For $X_{2i}$ and $X_{2e}, v \neq 5$; for $Y_{14}$ and $Y_{1e}, v \notin \{5, 6\}$; for $Z_{14}$ and $Z_{1e}, v \neq 5$; and for $Z_{2i}$ and $Z_{2e}, v \neq 5$.

Proof. The necessary conditions follow from Lemmas 2.1 and 2.2. We now establish sufficiency in several cases. In cases 1–8, reduce vertex labels modulo $v$ when $v \equiv 1$ (mod 5) and reduce vertex labels modulo $v - 1$ when $v \equiv 0$ (mod 5).

Case 1. For a $C_{4i}$ decomposition of $D_v$ for $v = 5\ell$, consider: $\{[j, 1 + 2i + j, 5\ell - 2 + j, 5\ell - 3 - 2i + j; 3\ell - 2 - i + j]_{C_{4i}} | i = 0, 1, \ldots, \ell - 2, j = 0, 1, \ldots, 5\ell - 2\} \cup \{[j, 2\ell + j, \infty, 3\ell + j; 2\ell - 1 + j]_{C_{4i}} | j = 0, 1, \ldots, 5\ell - 2\}.

For a $C_{4i}$ decomposition of $D_v$ for $v = 5\ell + 1$, consider: $\{[j, 1 + 2i + j, 5\ell + j, 5\ell - 1 - 2i + j; 2\ell + 1 + i + j]_{C_{4i}} | i = 0, 1, \ldots, \ell - 1, j = 0, 1, \ldots, 5\ell\}.

Case 2. For an $X_{1i}$ decomposition of $D_v$ for $v = 5\ell$, consider: $\{[j, \ell + i + j, \ell - 1 + j, 4\ell - 2 - i + j; 5\ell - 2 - i + j]_{C_{4i}} | i = 0, 1, \ldots, \ell - 2, j = 0, 1, \ldots, 5\ell\}.$
0, 1, ..., 5\ell - 2 \} \cup \{ [j, 2\ell - 1 + j, \infty, 3\ell - 1 + j; 4\ell - 1 + j]_{C_{4i}} \mid j = 0, 1, \ldots, 5\ell - 2 \}.$

For an $X_{1i}$ decomposition of $D_v$ for $v = 10\ell + 1$, consider: $\{ [j, 2 + 5i + j, 4 + 10i + j, 3 + 5i + j; 10\ell - 10i + j]_{X_{1i}} \mid i = 0, 1, \ldots, \ell - 1, j = 0, 1, \ldots, 10\ell \} \cup \{ [j, 10\ell - 3 + 5i + j, 10\ell + j, 10\ell - 4 - 5i + j; 10\ell - 5 + 10i + j]_{X_{1i}} \mid i = 0, 1, \ldots, \ell - 1, j = 0, 1, \ldots, 10\ell \}.$

For an $X_{1i}$ decomposition of $D_v$ for $v = 10\ell + 6$, consider: $\{ [j, 2 + 5i + j, 4 + 10i + j, 3 + 5i + j; 10\ell + 5 + 10i + j]_{X_{1i}} \mid i = 0, 1, \ldots, \ell, j = 0, 1, \ldots, 10\ell + 5 \} \cup \{ [j, 10\ell + 2 - 5i + j, 10\ell + 5 + j, 10\ell + 1 - 5i + j; 10\ell + 10i + j]_{X_{1i}} \mid i = 0, 1, \ldots, \ell - 1, j = 0, 1, \ldots, 10\ell + 5 \}.$

**Case 3.** For an $X_{2i}$ decomposition of $D_v$ for $v = 5\ell$, $\ell \geq 2$, consider: $\{ [j, 1 + i + j, 4\ell + 1 + i + j; \ell - i + j]_{X_{2i}} \mid i = 0, 1, \ldots, \ell - 2, j = 0, 1, \ldots, 5\ell - 2 \} \cup \{ [j, 2\ell - 1 + j, 4\ell - 1 + j, \infty; 1 + j]_{X_{2i}} \mid j = 0, 1, \ldots, 5\ell - 2 \}.$

For an $X_{2i}$ decomposition of $D_v$ for $v = 10\ell + 1$, consider: $\{ [j, 10\ell - 1 - 5i + j, 10\ell - 3 - 10i + j, 10\ell - 2 - 5i + j; 1 + 10i + j]_{X_{2i}} \mid i = 0, 1, \ldots, \ell - 1, j = 0, 1, \ldots, 10\ell \} \cup \{ [j, 4 + 5i + j, 8 + 10i + j, 3 + 5i + j; 6 + 10i + j]_{X_{2i}} \mid i = 0, 1, \ldots, \ell - 1, j = 0, 1, \ldots, 10\ell \}.$

For an $X_{2i}$ decomposition of $D_v$ for $v = 10\ell + 6$, consider: $\{ [j, 10\ell + 4 - 5i + j, 10\ell + 2 - 10i + j, 10\ell + 3 - 5i + j; 1 + 10i + j]_{X_{2i}} \mid i = 0, 1, \ldots, \ell, j = 0, 1, \ldots, 10\ell + 5 \} \cup \{ [j, 4 + 5i + j, 8 + 10i + j, 3 + 5i + j; 6 + 10i + j]_{X_{2i}} \mid i = 0, 1, \ldots, \ell - 1, j = 0, 1, \ldots, 10\ell + 5 \}.$

**Case 4.** For an $X_{3i}$ decomposition of $D_v$ for $v = 5\ell$, consider: If $\ell = 1$ then take $\{ [j, \infty, 1 + j, 2 + j; 3 + j]_{X_{3i}} \mid j = 0, 1, 2, 3 \}$ and if $\ell \geq 2$ then take $\{ [j, 2\ell + 1 + j, \ell - 1 + j, 5\ell - 2 - i + j; \ell - i + j]_{X_{3i}} \mid i = 0, 1, \ldots, \ell - 3, j = 0, 1, \ldots, 5\ell - 2 \} \cup \{ [j, 3\ell - 2 + j, \ell - 1 + j, 4\ell + j; 3\ell + j]_{X_{3i}} \mid j = 0, 1, \ldots, 5\ell - 2 \}.$

For an $X_{3i}$ decomposition of $D_v$ for $v = 10\ell + 1$, consider: $\{ [j, 1 + 5i + j, 4 + 10i + j, 2 + 5i + j; 10\ell - 4 - 10i + j]_{X_{3i}} \mid i = 0, 1, \ldots, \ell - 1, j = 0, 1, \ldots, 10\ell \} \cup \{ [j, 10\ell - 4 - 5i + j, 10\ell - 7 - 10i + j, 10\ell - 3 - 5i + j; 10\ell - 9 - 10i + j]_{X_{3i}} \mid i = 0, 1, \ldots, \ell - 1, j = 0, 1, \ldots, 10\ell \}.$

For an $X_{3i}$ decomposition of $D_v$ for $v = 10\ell + 6$, consider: $\{ [j, 1 + 5i + j, 4 + 10i + j, 2 + 5i + j; 10\ell + 1 - 10i + j]_{X_{3i}} \mid i = 0, 1, \ldots, \ell, j = 0, 1, \ldots, 10\ell + 5 \} \cup \{ [j, 10\ell + 1 - 5i + j, 10\ell - 2 - 10i + j, 10\ell + 2 - 5i + j; 10\ell - 4 - 10i + j]_{X_{3i}} \mid i = 0, 1, \ldots, \ell - 1, j = 0, 1, \ldots, 10\ell + 5 \}.$

**Case 5.** For an $X_{4i}$ decomposition of $D_v$ for $v = 5\ell$, consider: If $\ell = 1$ then take $\{ [j, 3 + j, \infty, 1 + j; 2 + j]_{X_{4i}} \mid j = 0, 1, 2, 3 \}$, if $\ell = 2$ then take $\{ [j, 8 + j, 4 + j, 7 + j; 5 + j]_{X_{4i}} \mid j = 0, 1, \ldots, 8 \}$, and if $\ell \geq 3$ then take $\{ [j, \ell + 1 + i + j, \ell + j, 4\ell - 1 - i + j; 5\ell - 2 - i + j]_{X_{4i}} \mid j = 0, 1, \ldots, 10\ell + 5 \}.$
$i = 0, 1, \ldots, \ell - 3, j = 0, 1, \ldots, 5\ell - 2 \cup \{ [j, 4\ell + j, \ell + 2 + j, 3\ell + 1 + j; 4\ell - 1 + j]_{X_{4i}}, [j, 2\ell + j, \infty, 3\ell + j; 4\ell + j]_{X_{4i}} | j = 0, 1, \ldots, 5\ell - 2 \}.$

For an $X_{4i}$ decomposition of $D_v$ for $v = 10\ell + 1$, consider: \{ [j, 2 + 5i + j, 1 + j, 10\ell - 1 - 5i + j; 10\ell - 10i + j]_{X_{4i}} | i = 0, 1, \ldots, \ell - 1, j = 0, 1, \ldots, 10\ell \} \cup \{ [j, 3 + 5i + j, 10\ell + j, 4 + 5i + j; 10\ell - 5 - 10i + j]_{X_{4i}} | i = 0, 1, \ldots, \ell - 1, j = 0, 1, \ldots, 10\ell \}.$

For an $X_{4i}$ decomposition of $D_v$ for $v = 10\ell + 6$, consider: \{ [j, 2 + 5i + j, 1 + j, 10\ell + 4 - 5i + j; 10\ell + 5 = 10i + j]_{X_{4i}} | i = 0, 1, \ldots, \ell, j = 0, 1, \ldots, 10\ell + 5 \} \cup \{ [j, 3 + 5i + j, 10\ell + 5 + j, 4 + 5i + j; 10\ell - 5 - 10i + j]_{X_{4i}} | i = 0, 1, \ldots, \ell - 1, j = 0, 1, \ldots, 10\ell \}.$

**Case 6.** For an $Y_{1i}$ decomposition of $D_v$ for $v = 5\ell, v \neq 5$, consider: \{ [j, 5\ell - 2 - 5i + j, 5\ell - 7 - 10i + j, 5\ell - 5 - 5i + j; 5\ell - 4 - 5i + j]_{Y_{1i}} | i = 0, 1, \ldots, \ell - 2, j = 0, 1, \ldots, 5\ell - 2 \} \cup \{ [j, 5\ell - j, 3 + j; 2 + j]_{Y_{1i}} | j = 0, 1, \ldots, 5\ell - 2 \}.$

For a $Y_{1i}$ decomposition of $D_v$ for $v = 5\ell + 1, v \neq 6$, consider: \{ [j, 5\ell - 1 - 5i + j, 5\ell - 6 - 10i + j, 5\ell - 3 - 5i + j; 5\ell - 5 - 5i + j]_{Y_{1i}} | i = 0, 1, \ldots, \ell - 3, j = 0, 1, \ldots, 5\ell \} \cup \{ [j, 4 + j, 5 + j, 2 + j; 10 + j]_{Y_{1i}} | j = 0, 1, \ldots, 5\ell \} \cup \{ [j, 9 + j, 15 + j, 7 + j; 5 + j]_{Y_{1i}} | j = 0, 1, \ldots, 5\ell \}.$

**Case 7.** For a $Y_{2i}$ decomposition of $D_v$ for $v = 5\ell$, consider: \{ [j, 1 + 5i + j, 5\ell - 2 + j, 5\ell - 6 - 5i + j; 5\ell - 4 - 5i + j]_{Y_{2i}} | i = 0, 1, \ldots, \ell - 2, j = 0, 1, \ldots, 5\ell - 2 \} \cup \{ [j, 5\ell - 3 + j, x; 1 + j]_{Y_{2i}} | j = 0, 1, \ldots, 5\ell - 2 \}.$

For a $Y_{2i}$ decomposition of $D_v$ for $v = 5\ell + 1$, consider: \{ [j, 2 + 5i + j, 5\ell - 1 + j, 1 + 5i + j; 5 + 5i + j]_{Y_{2i}} | i = 0, 1, \ldots, \ell - 1, j = 0, 1, \ldots, 5\ell \}.$

**Case 8.** For a $Y_{3i}$ decomposition of $D_v$ for $v = 5\ell$, consider: \{ [j, 2 + 5i + j, 7 + 10i + j, 4 + 5i + j; 3 + 5i + j]_{Y_{3i}} | i = 0, 1, \ldots, \ell - 2, j = 0, 1, \ldots, 5\ell - 2 \} \cup \{ [j, 5\ell - 3 + j, 5\ell - 4 + j, \infty; 5\ell - 2 + j]_{Y_{3i}} | j = 0, 1, \ldots, 5\ell - 2 \}.$

For a $Y_{3i}$ decomposition of $D_v$ for $v = 5\ell + 1$, consider: \{ [j, \ell + 1 + 2i + j, \ell + j, \ell + 2 + 2i + j; 5\ell - i + j]_{Y_{3i}} | i = 0, 1, \ldots, \ell - 1, j = 0, 1, \ldots, 5\ell \}.$

**Case 9.** For a $Z_{1i}$ decomposition of $D_v$ for $v \equiv 0 \pmod{5}$, we present a recursive argument. First, to show that a $Z_{1i}$ decomposition of $D_{10}$ exists, suppose the vertex set of $D_{10}$ is $\{0_1, 1_1, 2_1, 3_1, 4_1, 0_2, 1_2, 2_2, 3_2, 4_2\}$. Consider the blocks: $\{ [0_2, 0_1, 4_2, 2_1; 1_2]_{Z_{1i}}, [4_2, 2_2, 3_2, 1_2; 3_2]_{Z_{1i}}, [0_2, 3_1, 1_2, 4_1; 1_2]_{Z_{1i}}, [2_2, 2_1, 1_2, 3_2; 3_1]_{Z_{1i}}, [1_2, 0_2, 4_2, 3_2; 0_1]_{Z_{1i}}, [2_2, 0_1, 3_2, 2_1; 1_1]_{Z_{1i}}, [3_2, 1_2, 4_2, 1_2; 2_1]_{Z_{1i}}, [0_2, 2_2, 1_2, 4_2; 3_2]_{Z_{1i}}, [2_2, 0_2, 3_2, 4_2; 1_2]_{Z_{1i}}, [0_2, 0_1, 4_1, 2_1; 1_2]_{Z_{1i}}, [4_1, 2_1, 3_1, 1_1; 3_2]_{Z_{1i}}, [0_1, 3_2, 1_2, 1_2; 4_2]_{Z_{1i}}, [2_1, 2_2, 1_1, 4_2; 3_1]_{Z_{1i}}, [1_1, 0_1, 4_1, 3_1; 0_2]_{Z_{1i}}, [2_1, 0_2, 3_1, 3_2; 1_2]_{Z_{1i}}, [3_1, 1_2, 4_1, 4_2; 2_2]_{Z_{1i}}, [0_1, 2_1, 1_1, 4_1; 3_1]_{Z_{1i}}, [2_1, 0_1, 3_1, 4_1; 1_1]_{Z_{1i}}.$

We now show that $Z_{1i}$ decomposition of $D_{5\ell}$ exists for all $\ell \geq 2$. We do so recursively and establish this result by induction. We have seen
that there is a decomposition for \( \ell = 2 \). Now suppose that such a decomposition exists for \( \ell = k \). Then there exists a \( Z_{1i} \) decomposition of \( D_{5k} \) where the vertex set of \( D_{5k} \) is \( V = \{0_1, 1_1, 2_1, 3_1, 4_1, 0_2, 1_2, 2_2, 3_2, 4_2, \ldots, 0_k, 1_k, 2_k, 3_k, 4_k\} \). Consider \( D_{5k+5} \) with vertex set \( V \cup \{0_{k+1}, 1_{k+1}, 2_{k+1}, 3_{k+1}, 4_{k+1}\} \). Add to the decomposition of \( D_{5k} \) the following: \( \{0_k, 1_1, 2_k, 3_1, 4_1, 0_2, 1_2, 2_2, 3_2, 4_2, \ldots, 0_k, 1_k, 2_k, 3_k, 4_k\} \). Consider \( D_{5k+5} \) with vertex set \( V \cup \{0_{k+1}, 1_{k+1}, 2_{k+1}, 3_{k+1}, 4_{k+1}\} \). Add to the decomposition of \( D_{5k} \) the following: \( \{0_k, 1_1, 2_k, 3_1, 4_1, 0_2, 1_2, 2_2, 3_2, 4_2, \ldots, 0_k, 1_k, 2_k, 3_k, 4_k\} \).

For a \( Z_{1i} \) decomposition of \( D_v \) for \( v = 5\ell + 1 \), consider: \( \{j, 2 + 5i + j, 5\ell - 1 + j, 3 + 5i + j, 1 + 5i + j\} \). Here, reduce vertex labels modulo \( v \).

**Case 10.** For a \( Z_{2i} \) decomposition of \( D_v \) for \( v \equiv 0 \pmod{5} \), we present a recursive argument. First, to show that a \( Z_{2i} \) decomposition of \( D_{10} \) exists, suppose the vertex set of \( D_{10} \) is \( \{0_1, 1_1, 2_1, 3_1, 4_1, 0_2, 1_2, 2_2, 3_2, 4_2\} \). Consider the blocks: \( \{1_2, 0_2, 3_2, 4_2\} Z_{2i}, \{0_2, 1_2, 2_2, 3_2\} Z_{2i}, \{1_2, 2_2, 3_2, 4_2\} Z_{2i}, \{0_2, 1_2, 2_2, 3_2\} Z_{2i}, \{1_2, 2_2, 3_2, 4_2\} Z_{2i}, \{0_2, 1_2, 2_2, 3_2\} Z_{2i}, \{1_2, 2_2, 3_2, 4_2\} Z_{2i}, \{0_2, 1_2, 2_2, 3_2\} Z_{2i}, \{1_2, 2_2, 3_2, 4_2\} Z_{2i}, \{0_2, 1_2, 2_2, 3_2\} Z_{2i}, \{1_2, 2_2, 3_2, 4_2\} Z_{2i}, \{0_2, 1_2, 2_2, 3_2\} Z_{2i}, \{1_2, 2_2, 3_2, 4_2\} Z_{2i}, \{0_2, 1_2, 2_2, 3_2\} Z_{2i}, \{1_2, 2_2, 3_2, 4_2\} Z_{2i}, \{0_2, 1_2, 2_2, 3_2\} Z_{2i}, \{1_2, 2_2, 3_2, 4_2\} Z_{2i}, \{0_2, 1_2, 2_2, 3_2\} Z_{2i}, \{1_2, 2_2, 3_2, 4_2\} Z_{2i} \}. \)

We now show that \( Z_{2i} \) decomposition of \( D_{5\ell} \) exists for all \( \ell \geq 2 \). We do so recursively and establish this result by induction. We have seen that there is a decomposition for \( \ell = 2 \). Now suppose that such a decomposition exists for \( \ell = k \). Then there exists a \( Z_{2i} \) decomposition of \( D_{5k} \) where the vertex set of \( D_{5k} \) is \( V = \{0_1, 1_1, 2_1, 3_1, 4_1, 0_2, 1_2, 2_2, 3_2, 4_2, \ldots, 0_k, 1_k, 2_k, 3_k, 4_k\} \). Consider \( D_{5k+5} \) with vertex set \( V \cup \{0_{k+1}, 1_{k+1}, 2_{k+1}, 3_{k+1}, 4_{k+1}\} \). Add to the decomposition of \( D_{5k} \) the following: \( \{1_1, 0_k, 3_k, 4_k, 1_2, 2_k, 3_k, 4_k, 1_2, 2_k, 3_k, 4_k, 1_2, 2_k, 3_k, 4_k\} Z_{2i}, \{0_k, 1_1, 2_1, 3_1, 4_1, 0_2, 1_2, 2_2, 3_2, 4_2, \ldots, 0_k, 1_k, 2_k, 3_k, 4_k\} \). Consider \( D_{5k+5} \) with vertex set \( V \cup \{0_{k+1}, 1_{k+1}, 2_{k+1}, 3_{k+1}, 4_{k+1}\} \). Add to the decomposition of \( D_{5k} \) the following: \( \{1_1, 0_k, 3_k, 4_k, 1_2, 2_k, 3_k, 4_k, 1_2, 2_k, 3_k, 4_k, 1_2, 2_k, 3_k, 4_k\} Z_{2i}, \{0_k, 1_1, 2_1, 3_1, 4_1, 0_2, 1_2, 2_2, 3_2, 4_2, \ldots, 0_k, 1_k, 2_k, 3_k, 4_k\} \).

For a \( Z_{2i} \) decomposition of \( D_v \) for \( v = 5\ell + 1 \), consider: \( \{j, 5\ell - 1 - 5i + j, 2 + j, 5\ell - 2 - 5i + j, 5\ell - 4 - 5i + j\} Z_{2i} \). Here, reduce vertex labels modulo \( v \).
In each case, the presented blocks form a decomposition of $D_v$. Corresponding decompositions of $D_v$ into each of the converses of the digraphs of Cases 1–10 immediately follow.

References


Appendix

In this appendix, we give an argument that no $X_{2i}$ decomposition of $D_5$ exists. Notice that the only vertex of $[a, b, c, d; e]_{X_{2i}}$ which is of odd indegree is $d$. So if $B_1 = [a_1, b_1, c_1, x; e_1]_{X_{2i}}$ is a block of such a decomposition
then, since vertex \( x \) has in-degree 4 (even) in \( D_5 \), another block of the decomposition must be of the form \( B_2 = [a_2, \ b_2, \ c_2, \ x; \ e_2]_{X_{2i}} \neq B_1 \). Now let the vertices of \( D_5 \) be \( v_1, \ v_2, \ v_3, \ v_4, \ v_5 \) and let the blocks of a decomposition be \( B_1, B_2, B_3, B_4 \). Without loss of generality, \( B_1 = [v_1, \ v_2, \ v_3, \ v_4; \ v_5]_{X_{2i}} \). By the above argument, we know that another block of the decomposition (say block \( B_2 \)) is of the form \( B_2 = [a_2, \ b_2, \ c_2, \ v_4; \ e_2]_{X_{2i}} \) where \( \{a_2, b_2, c_2, e_2\} = \{v_1, v_2, v_3, v_5\} \). Notice that there are 4! ways to assign vertices \( v_1, v_2, v_3, v_5 \) to the positions \( a_2, b_2, c_2, e_2 \) in block \( B_2 \). Now if block \( B_3 \) is of the form \( [a_3, b_3, c_3, v_i; e_3]_{X_{2i}} \) then block \( B_4 \) must be of the form \( [a_4, b_4, c_4, v_i; e_4]_{X_{2i}} \), again by the above argument. There are 5 ways to assign a vertex to vertex \( v_i \) in blocks \( B_3 \) and \( B_4 \). There are then 4! ways to assign the remaining vertices to positions \( a_3, b_3, c_3, e_3 \) in block \( B_3 \) and 4! ways to assign the remaining vertices to positions \( a_4, b_4, c_4, e_4 \) in block \( B_4 \). Hence, there are \( 4! \times 4! \times 4! \times 5 = 69,120 \) different ways to assign the 5 vertices of to blocks \( B_1, B_2, B_3, B_4 \). If such a decomposition exists, then it must be one of these possibilities. The authors have written a program to test these possibilities and none of them yield a decomposition. (The authors readily admit that there must be a more elegant argument for this—\( D_5 \) and \( X_{2i} \) each contain 5 vertices, so certainly reducing this to 69,120 cases and then beating them death cannot be the best method!)