# Cyclic, $f$-Cyclic, and Bicyclic Decompositions of the Complete Graph into the 4-Cycle with a Pendant Edge

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**Abstract.** In this paper, we consider decompositions of the complete graph on $v$ vertices into 4-cycles with a pendant edge which admit automorphisms consisting of: (1) a single cycle of length $v$, (2) $f$ fixed points and a cycle of length $v-f$, or (3) two disjoint cycles.

## 1 Introduction

A *$g$-decomposition of graph $G$* is a set of subgraphs of $G$, $\gamma = \{g_1, g_2, \ldots, g_n\}$, where $g_i \cong g$ for $i \in \{1, 2, \ldots, n\}$, $E(g_i) \cap E(g_j) = \emptyset$ for $i \neq j$, and $\bigcup_{i=1}^{n} E(g_i) = E(G)$. The $g_i$ are called *blocks* of the decomposition. The study of graph decompositions is a vibrant area of research [1, 4, 5]. Of relevance to our study are decompositions of $K_v$. In particular, a 3-cycle ($C_3$) decomposition of $K_v$ exists if and only if $v \equiv 1$ or $3 \pmod 6$ and such a structure is called a *Steiner triple system* [9]. It is well known that a $C_4$-decomposition of $K_v$ exists if and only if $v \equiv 1 \pmod 8$. Let $L$ denote the graph $C_3 \cup \{e\}$ (that is, $V(L) = \{a, b, c, d\}$ and $E(L) = \{(a, b), (b, c), (a, c), (a, d)\}$, the 3-cycle with a pendant edge). An $L$-decomposition of $K_v$ exists if and only if $v \equiv 0$ or $1 \pmod 8$ [3]. Let $H$ denote the graph $C_4 \cup \{e\}$ (that is, $V(H) = \{a, b, c, d, e\}$ and $E(H) = \{(a, b), (b, c), (c, d), (a, d), (a, e)\}$; we denote such $H$ as $[a, b, c, d; e]$), the 4-cycle with a pendant edge. See Figure 1. An $H$-decomposition of $K_v$ exists if and only if $v \equiv 0$ or $1 \pmod 5$, $v \geq 10$ [2].

An *automorphism* of a $g$-decomposition of $G$ is a permutation $\pi$ of $V(G)$ which fixes set $\gamma$. The *orbit* of a block $g_i$ under $\pi$ is the set $\{\pi^n(g_i) \mid n \in \mathbb{N}\}$ and the *length* of the orbit of $g_i$ is the cardinality of the orbit of $g_i$. A set $B$ of blocks is a set of *base blocks* under permutation $\pi$ if the orbits of the blocks of $B$ generate an $H$-decomposition of $K_v$ and the orbits of the elements of $B$ are disjoint. An automorphism is said to be *cyclic* if it consists of a single cycle, is said to be *$f$-cyclic* if it consists of $f$ fixed points and a single cycle, and is said to be *bicyclic* if it consists of two disjoint cycles. A common method of construction for graph decompositions is the use of difference methods and cyclic permutations. A cyclic $C_3$-decomposition of $K_v$ exists

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if and only if $v \equiv 1$ or $3 \pmod 6$, $v \neq 9$ [11]. It is well known that a cyclic $C_4$-decomposition of $K_v$ exists if and only if $v \equiv 1 \pmod 8$. A cyclic $C_3$-decomposition of $K_v$ exists if and only if $v \equiv 1$ or $3 \pmod 6$, $v \neq 9$ [11]. It is well known that a cyclic $C_4$-decomposition of $K_v$ exists if and only if $v \equiv 1 \pmod 8$. A cyclic $L$-decomposition of $K_v$ exists if and only if $v \equiv 1 \pmod 8$ [3, 8]. The $f$-cyclic automorphism was introduced by Micale and Pennisi in connection with oriented triple systems, which concern decompositions of complete symmetric digraphs into orientations of a 3-cycle [10]. When discussing bicyclic automorphisms, we assume that the cycles have lengths $M$ and $N$ where $M \leq N$. A bicyclic $C_3$-decomposition of $K_v$ exists if and only if $v = M+N \equiv 1$ or $3 \pmod 6$, $M \equiv 1$ or $3 \pmod 6$, $M \neq 9$ $(M > 1)$, and $M \mid N$ [6]. A bicyclic $L$-decomposition of $K_v$ exists if and only if (*i*) $N = 2M$ and $v = M + N \equiv 9 \pmod{24}$, or (*ii*) $m \equiv 1 \pmod 8$ and $N = kM$ where $k \equiv 7 \pmod 8$ [8]. The purpose of this paper is to give necessary and sufficient conditions for the existence of cyclic, $f$-cyclic, and bicyclic $H$-decompositions of $K_v$.

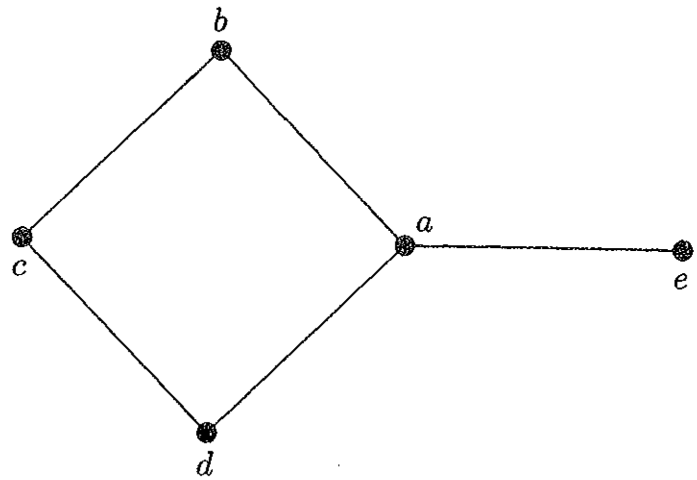


**Figure 1.** We denote this graph as $H = [a, b, c, d; e]$.

## 2 Cyclic and Rotational $H$-Decompositions

The following result gives necessary and sufficient conditions for the existence of a cyclic $H$-decomposition of $K_v$.

**Theorem 2.1** *A cyclic $H$-decomposition of $K_v$ exists if and only if $v \equiv 1$ (mod 10).*

**Proof.** We consider cyclic $H$-decompositions of $K_v$ where $V(K_v) = \{0, 1, 2,$

$\ldots, (v-1)\}$ and where the cyclic permutation is $\pi = (0, 1, 2, \ldots, v-1)$.

Suppose such a system exists for $v \equiv 0$ or $6 \pmod{10}$. Then by raising $\pi$ to the $v/2$ power, we see that the edge $(0, v/2)$ is fixed by interchanging the two vertices $0$ and $v/2$. Since the edge $(0, v/2)$ is in exactly one copy of $H$ in the decomposition, then this copy of $H$ must be fixed by $\pi^{v/2}$. However, it is not possible to fix $H$ with a permutation which interchanges the ends of an edge. Therefore such systems do not exist.

Now suppose $v \equiv 5 \pmod{10}$. The length of the orbit of each edge and every block $g_i$ of set $\gamma$ is $v$. Therefore the orbits of the $g_i$ create a partition of $\gamma$ into $|\gamma|/v$ sets. But with $v \equiv 5 \pmod{10}$, $v$ does not divide $|\gamma|$ and so such a system does not exist.

Suppose $v \equiv 1 \pmod{10}$, say $v = 10k+1$. If $v = 11$, consider $\{[0, 1, 5, 3; 6]\}$. If $v = 21$, consider $\{[0, 1, 5, 3; 6], [0, 7, 17, 9; 5]\}$. If $v \geq 31$, consider $\{[0, 1, 5, 3; 6], [0, 7, 17, 9; 5]\} \bigcup \{[0, 5i+11, 10i+25, 5i+13; 5i+15] \mid i = 0, 1, \ldots, k-3\}$. In each case, a set of base blocks is given for a cyclic $H$-decomposition of $K_v$ under $\pi$. Here, and throughout, we assume that vertex labels are reduced modulo the length of the cycle containing them. ■

It has recently come to our attention that the sufficiency of Theorem 2.1 can be established by considering vertex labelings. Let $V(H) = V_1 \cup V_2$ where sets $V_1$ and $V_2$ form the bipartition of $H$, $|V_1| = 2$, and $|V_2| = 3$. An $\alpha$-labeling of graph $H$ is a one-to-one function $f : V(H) \to \{0, 1, 2, 3, 4, 5\}$ if $\{|f(u) - f(v)| \mid (u, v) \in E(H)\} = \{1, 2, 3, 4, 5\}$ for which there exists integer $k$ such that $f(v_1) \leq k$ for $v_1 \in V_1$ and $f(v_2) > k$ for $v_2 \in V_2$. An $\alpha$-labeling of $H$ is given by labeling the vertices of $H$ as given in Figure 1 as $f(a) = 0$, $f(b) = 2$, $f(c) = 1$, $f(d) = 5$, and $f(e) = 3$. Rosa (see [7] and [12]) showed that if a graph $G$ with $n$ edges admits an $\alpha$-labeling, then there exists a cyclic $G$-decomposition of $K_{2nx+1}$ for all natural numbers $x$. This result, therefore, gives an alternate proof of sufficiency in Theorem 2.1.

A special case of a bicyclic permutation is a permutation consisting of a single fixed point and a single cycle ($M = 1$ and $N = v - 1$ in the notation of Section 1). A graph decomposition admitting such a partition is said to be *rotational* (or *1-rotational*). The following theorem classifies rotational $H$-decompositions of $K_v$.

**Theorem 2.2** *A rotational $H$-decomposition of $K_v$ exists if and only if $v \equiv 0 \pmod{10}$.*

**Proof.** In such a system, the length of the orbit of each block is $v - 1$. Therefore the number of edges must be a multiple of $5(v-1)$. Now

$|E(K_v)| = \frac{v(v-1)}{2}$, so it follows that $v \equiv 0 \pmod{10}$ is necessary. So suppose $v \equiv 0 \pmod{10}$, say $v = 10k$, $V(K_v) = \{\infty, 0, 1, 2, \ldots, (v-1)\}$ and $\pi = (\infty)(0, 1, 2, \cdots, (v-1))$. Consider the set of blocks:

$$\{[0,1,5,3;\infty]\} \bigcup \{[0, 5i+5, 10i+13, 5i+7; 5i+9] \mid i = 0, 1, \ldots, k-2\}.$$

This is a set of base blocks for a rotational $H$-decomposition of $K_v$ under $\pi$. ∎

# 3 The $f$-Cyclic Results

We now consider a permutation of an $H$-decomposition of $K_v$ where the permutation consists of $f$ fixed points and a cycle of length $v - f$.

**Lemma 3.1.** *The fixed points of an $f$-cyclic automorphism of an $H$-decomposition of $K_v$ form a subsystem. That is, if $\pi$ is the $f$-cyclic automorphism, $(a, b)$ is an edge of a block $h$, and $\pi(a) = a$, $\pi(b) = b$, then each vertex of $h$ is fixed by $\pi$.*

**Proof.** Edge $(a, b)$ appears in exactly one block of a decomposition. Since $(a, b)$ is in both $h$ and $\pi(h)$, it must be that $h = \pi(h)$. The only way to fix an edge of $h$ without fixing all vertices of $h$ is to fix three vertices of $h$ and interchange the other two. If this is the case, then $\pi$ must consist of several (at least three) fixed points and a transposition. Say the vertices in the transposition are $c$ and $d$. Edge $(c, d)$ must be in some block, $h'$. But $\pi$ fixes edge $(c, d)$ and hence must fix block $h'$. However, it is impossible to fix $H$ while interchanging the vertices of one of its edges. Therefore $\pi$ cannot consist of fixed points and a transposition and it must be that $\pi$ fixes all vertices of $h$. ∎

We let the vertex set of $K_v$ be $\{\infty_1, \infty_2 \ldots, \infty_f\} \cup \{0, 1, \ldots, v - f - 1\}$ and the $f$-cyclic permutation be $(\infty_1)(\infty_2)\cdots(\infty_f)(0, 1, \ldots (v - f - 1))$. Lemma 3.1 along with the necessary condition for the existence of an $H$-decomposition of $K_v$ implies the following.

**Lemma 3.2.** *In an $f$-cyclic $H$-decomposition of $K_v$, it is necessary that $f \equiv 0$ or $1 \pmod{5}$, $f \geq 10$.*

**Lemma 3.3.** *In an $f$-cyclic $H$-decomposition of $K_v$, it is necessary that $f \leq (v-1)/9$.*

**Proof.** Suppose a block of such a decomposition contains edges of the forms $(\infty_i, a)$ and $(\infty_i, b)$ where $a, b \in \mathbb{Z}_{v-f}$ with $a > b$. Then $\pi^{b-a}$ maps edge $(\infty_i, a)$ to $(\infty_i, b)$. Since $(\infty_i, b)$ occurs in only one block, $\pi^{b-a}$ must fix this block. But the only way to fix $H$ without fixing each vertex is to fix three of the vertices of $H$ and interchange the other two. So $\pi^{b-a}$ must consist of fixed points and transpositions. However, the pendant edge must be fixed by $\pi^{b-a}$ and this can occur only if both vertices of the pendant edge are fixed. But this contradicts Lemma 3.1. Therefore no block of an $f$-cyclic $H$-decomposition may include edges of the forms $(\infty_i, a)$ and $(\infty_i, b)$ where $a, b \in \mathbb{Z}_{v-f}$.

Again, by Lemma 3.1, we see that the admissible blocks of such a decomposition must be of the following forms only: $B_\infty = [\infty_i, \infty_j, \infty_k, \infty_l; \infty_m]$, $B_{C\infty} = [a, b, c, d; \infty_i]$, and $B_C = [a, b, c, d; e]$ where $a, b, c, d \in \mathbb{Z}_{v-f}$. Block $B_\infty$ is fixed by $\pi$ and all blocks of this form make up an $H$-decomposition of $K_f$. So there are $f(f-1)/10$ such blocks. The length of the orbit of a block of type $B_{C\infty}$ is $v - f$ and the orbit of this block contains all edges of the form $(\infty_i, a)$ for fixed $i$ and any $a \in \mathbb{Z}_{v-f}$. Therefore there must be $f(v-f)$ blocks of this form. These blocks contain $4f(v-f)$ edges of the form $(a, b)$ where $a, b \in \mathbb{Z}_{v-f}$. Since $K_v$ has $(v-f)(v-f-1)/2$ such edges, it is necessary that $4f(v-f) \le (v-f)(v-f-1)/2$, or $f \le (v-1)/9$. ∎

**Lemma 3.4.** *The following conditions are necessary for the existence of an $f$-cyclic $H$-decomposition of $K_v$:*
(1) *if $v \equiv 0 \pmod{10}$, then $f \equiv 1 \pmod{10}$, or*
(2) *if $v \equiv 1 \pmod{10}$, then $f \equiv 0 \pmod{10}$, or*
(3) *if $v \equiv 5 \pmod{10}$, then $f \equiv 6 \pmod{10}$, or*
(4) *if $v \equiv 6 \pmod{10}$, then $f \equiv 5 \pmod{10}$.*

**Proof.** With the notation of Lemma 3.3, the number of edges of the form $(a, b)$, where $a \in \mathbb{Z}_{v-f}$, which are *not* in blocks of the form $B_{C\infty}$ is

$$\frac{(v-f)(v-f-1)}{2} - 4f(v-f) = (v-f)\left(\frac{v-9f-1}{2}\right).$$

These edges must be contained in blocks of the form $B_C$. Since each such block contains five such edges, there must by $(v-f)(v-9f-1)/10$ such blocks. The lengths of the orbit of each $B_C$ is $v - f$, and so there must be $(v-9f-1)/10$ base blocks of the form $B_C$. Since $v \equiv 0$ or $1 \pmod 5$ and $f \equiv 0$ or $1 \pmod 5$, the conditions on $v$ and $f$ follow. ∎

**Theorem 3.5** *An $f$-cyclic $H$-decomposition of $K_v$ exists if and only if $f \le (v-1)/9$ and*
(1) *if $v \equiv 0 \pmod{10}$, then $f \equiv 1 \pmod{10}$, or*

(2) *if* $v \equiv 1 \pmod{10}$, *then* $f \equiv 0 \pmod{10}$, *or*
(3) *if* $v \equiv 5 \pmod{10}$, *then* $f \equiv 6 \pmod{10}$, *or*
(4) *if* $v \equiv 6 \pmod{10}$, *then* $f \equiv 5 \pmod{10}$.

**Proof.** The necessary conditions follow from Lemmas 3.3 and 3.4. For sufficiency, consider the set

$$\{[0, 1+4i, 5+8i, 3+4i; \infty_{i+1}] \mid i = 0,1,2,\ldots,f-1\} \cup \{[0, 4f+1+5i,$$

$$8f+5+10i, 4f+3+5i; 4f+5+5i] \mid i = 0,1,2,\ldots,(v-9f-11)/10\}.$$

This set, along with a set of blocks for an $H$-decomposition of $K_f$ on vertex set $\{\infty_1, \infty_2, \ldots, \infty_f\}$, forms a set of base blocks for an $f$-cyclic $H$-decomposition of $K_v$ for the necessary values of $v$ and $f$. ∎

# 4 The Bicyclic Results

In this section we consider bicyclic $H$-decompositions of $K_v$ where the vertex set of $K_v$ is $\{0_1, 1_1, 2_1, \ldots, (M-1)_1, 0_2, 1_2, 2_2, \ldots, (N-1)_2\}$ and the automorphism is $(0_1, 1_1, 2_1, \cdots, (M-1)_1)(0_2, 1_2, 2_2, \cdots, (N-1)_2)$. An argument similar to that used in the proof of Theorem 2.1 can be used to show that in a bicyclic automorphism, neither $M$ nor $N$ can be even (or there is the same uniqueness problem with edge $(0, M/2)$ or edge $(0, N/2)$, respectively). Therefore we have the following result.

**Lemma 4.1** *In a bicyclic $H$-decomposition of $K_v$, neither $M$ nor $N$ can be even.*

**Lemma 4.2** *If a bicyclic $H$-decomposition of $K_v$ exists where $M < N$, then $M \equiv 1 \pmod{10}$.*

**Proof.** Suppose a bicyclic $H$-decomposition of $K_v$ exists where $M < N$ and let $\pi$ be the bicyclic automorphism. Assume that there is a block $h$ of the decomposition with vertex set $V(h) = \{v_1, w_1, x_i, y_j, z_k\}$ and edge set satisfying $(v_1, w_1) \in E(h)$. Then $\pi^M$ fixes edge $(v_1, w_1)$ and hence must fix $h$. The only way to fix $h = [a, b, c, d; e]$ without fixing all of the vertices is to fix the vertices $a$, $c$, and $e$ and to interchange vertices $b$ and $d$. Therefore such a $\pi$ satisfies the property that $\pi^M$ fixes three vertices of $h$, say $v_1$, $w_1$, and $x_1$, and interchanges the other two vertices, $y_2$ and $z_2$. In this case, $\pi^M$ must consist of $M$ fixed points and $N/2$ transpositions (and so $N = 2M$). However, as seen in Lemma 4.1, $N$ cannot be even and hence all vertices of $h$ must be fixed by $\pi^M$ and in fact $V(h) = \{v_1, w_1, x_1, y_1, z_1\}$. That is, if a block of a bicyclic decomposition has one edge with vertices in $\{0_1, 1_1, 2_1, \ldots, (M-1)_1\}$, then all vertices of the block lie in this set. As in Lemma 3.1, such blocks form a subsystem of the bicyclic decomposition. If we restrict $\pi$ to these blocks, we see that they form a cyclic $H$-decomposition of $K_M$ and by Theorem 2.1, $M \equiv 1 \pmod{10}$. ∎

**Lemma 4.3** *A bicyclic $H$-decomposition of $K_v$ admitting an automorphism consisting of two disjoint cycles of the same length exists if and only if $v \equiv 6$ (mod 20), $v \geq 26$.*

**Proof.** With $M = N$ and $v = 2M$, we have from Lemma 4.1 that a necessary condition is $v \equiv 2 \pmod 4$. Since $v \equiv 0$ or $1 \pmod 5$, it is necessary that $v \equiv 6$ or $10 \pmod{20}$. Now if $v \equiv 10 \pmod{20}$, then $M \equiv 5$ (mod 10) and the length of the orbit of each edge and every block $g_i \in \gamma$ is $M$. Therefore the orbits of the $g_i$ create a partition of $\gamma$ into $|\gamma|/M$ sets. But with $v \equiv 10 \pmod{20}$, $M = v/2$ does not divide $|\gamma|$ and so such a system does not exist.

Now suppose $M = v/2 \equiv 3 \pmod{10}$, say $M = 10k + 3$. Then consider the set:

$$\{[0_p, 1_p, 5_p, 3_p; 6_p] \mid p = 1, 2\} \cup \{[0_1, (5k+3)_2, 2_1, 5k+2_2; 0_2],$$

$$[0_1, (5k+5)_2, 6_1, (5k+4)_2; 5_1], [0_2, (5k+7)_1, 10_2, (5k+6)_1; 5_2]\} \cup$$

$$\{[0_p, (7+5i)_p, (17+10i)_p, (9+5i)_p; (11+5i)_p], [0_1, (5k+9+4i)_2, (14+8i)_1,$$

$$(5k+8+4i)_2; (1+i)_2], [0_1, (5k+11+4i)_2, (18+8i)_1, (5k+10+4i)_2; (10k+2-i)_2]$$

$$\mid p = 1, 2; i = 0, 1, \ldots, k-2\}.$$

This a set of base blocks for a bicyclic $H$-decomposition of $K_v$ as needed. ∎

**Lemma 4.4** *If a bicyclic $H$-decomposition of $K_v$ exists with $M < N$, then $M \equiv 1$ (mod 10) and $N = kM$ where $k \equiv 9$ (mod 10).*

**Proof.** By Lemma 4.2, $M \equiv 1 \pmod{10}$. Suppose all edges of the form $(x_1, y_2)$ are contained in blocks consisting only of such edges (a possibility since $H$ is bipartite). Then the blocks with vertices from $\{0_2, 1_2, 2_2, \ldots, (N-1)_2\}$ form a cyclic $H$-decomposition of $K_N$ and by Theorem 2.1, $N \equiv 1$ (mod 10). But then $v = M + N \equiv 2 \pmod{10}$ and since $v \equiv 0$ or $1$ (mod 5), this is impossible. Therefore if a bicyclic $H$-decomposition exists with $M < N$, then there must be some block $h$ which contains both edges of the form $(x_1, y_2)$ and $(y_2, z_2)$ (it follows from the proof of Lemma 4.2 that no block can contain both edges of the form $(x_1, y_1)$ and $(y_1, z_2)$). Now if we apply $\pi^N$ to such a block, the edge $(y_2, z_2)$ is fixed and therefore the block containing $(y_2, z_2)$ is fixed. As in Lemma 4.2, this can be accomplished by interchanging two of the other vertices of $h$, but this would require that $\pi^N$ contains $M/2$ transpositions, a contradiction. Therefore all vertices of $h$ must be fixed and, in particular, $x_1$ must be fixed. Therefore $M$ is a multiple of $N$: $N = kM$ for some positive integer $k$.

From Lemma 4.2, we see that every edge of the form $(x_1, y_1)$ is in a block of the form $[a_1, b_1, c_1, d_1; e_1]$. Any edge of the form $(x_1, y_2)$ or the form $(x_2, y_2)$ has an orbit of length $N$ and there are $MN + N(N-1)/2$ such edges. Therefore any block consisting of such edges also has an orbit of length $N$ and the total number of edges in this orbit is $5N$. This implies that $5N$ divides $MN + N(N-1)/2$, or that $M + (N-1)/2 = M + (kM-1)/2 \equiv 0 \pmod 5$, from which follows the result $k \equiv 9 \pmod{10}$. ∎

**Lemma 4.5** *A bicyclic $H$-decomposition of $K_v$ with $M < N$ exists if and only if $M \equiv 1 \pmod{10}$ and $N = kM$ where $k \equiv 9 \pmod{10}$.*

**Proof.** The necessary conditions follow from Lemma 4.4. The case when $M = 1$ follows from Theorem 2.2. For $M > 1$, consider the following collection of blocks:

$$\{[0_1, (4+5i)_2, 2_1, (3+5i)_2; (5+5i)_2] \mid i = 0, 1, \ldots, (M-6)/5\}$$

$$\cup\{[0_2, 4_2, 2_2, 3_2; 0_1]\} \cup \{[0_2, (5+5i)_2, (13+10i)_2, (7+5i)_2; (9+5i)_2] \mid$$
$$i = 0, 1, \ldots, (N-19)/10\}.$$

These blocks, along with the base blocks of a cyclic $H$-decomposition of $K_M$ on vertex set $\{0_1, 1_1, \ldots, (M-1)_1\}$, form a set of base blocks for a bicyclic $H$-decomposition of $K_v$ as needed. ∎

Lemmas 4.2 to 4.5 combine to give necessary and sufficient conditions for a bicyclic $H$-decomposition of $K_v$.

**Theorem 4.6** *A bicyclic $H$-decomposition of $K_v$, where the bicyclic automorphism consists of disjoint cycles of lengths $M$ and $N$ where $M \leq N$ exists if and only if*
**(i)** $M = N \equiv 3 \pmod{10}$, $M = N \geq 13$, *or*
**(ii)** $M \equiv 1 \pmod{10}$ *and* $N = kM$ *where* $k \equiv 9 \pmod{10}$.

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