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Research Article

Packings and Coverings of Various Complete Graphs with the 4-Cycle with a Pendant Edge

Brandon Coker,[1] Gary D. Coker,[2] Robert Gardner,[3] and Yan Xia[3]

[1] *Department of Computer Science, Francis Marion University, Florence, SC 29502, USA*
[2] *Institute of Mathematics and Physics, Aberystwyth University, Aberystwyth, Ceredigion SY23 3BZ, UK*
[3] *Department of Mathematics and Statistics, East Tennessee State University, Johnson City, TN 37614, USA*

Correspondence should be addressed to Robert Gardner; gardnerr@etsu.edu

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We consider the packings and coverings of complete graphs with isomorphic copies of the 4-cycle with a pendant edge. Necessary and sufficient conditions are given for such structures for (1) complete graphs K_v, (2) complete bipartite graphs $K_{m,n}$, and (3) complete graphs with a hole $K(v, w)$. In the last two cases, we address both restricted and unrestricted coverings.

1. Introduction, Motivation, and History

A *g-decomposition of graph G* is a set of subgraphs of G, $\gamma = \{g_1, g_2, \dots, g_n\}$, where $g_i \cong g$ for $i \in \{1, 2, \dots, n\}$, $E(g_i) \cap E(g_j) = \varnothing$ for $i \neq j$, and $\cup_{i=1}^{n} E(g_i) = E(G)$. The g_i are called *blocks* of the decomposition. The concept of a graph decomposition lies in the general area of the *design theory*. We can relate a graph decomposition to an experimental design by considering the following hypothetical situation: "suppose you have a collection of v samples and you wish to compare a property of the samples. However, the only way to compare the samples is to run them three at a time in a machine which performs the comparison. The machine cannot be calibrated from run to run and so to compare two samples, we must run them together in the machine. When can all of the v samples be optimally compared to each other by running the machine $\binom{v}{2}/3$ times?" The solution to this question is equivalent to finding a K_3-decomposition of K_v, where each vertex of K_v represents a sample, an edge joining two vertices represents a comparison of the two corresponding samples, and a copy of K_3 represents a run of the machine. A K_3-decomposition of K_v exists if and only if $v \equiv 1$ or $3 \pmod 6$, and such a structure is called a *Steiner triple system* [1].

In the event that a g-decomposition of G does not exist, we can still consider a set of isomorphic copies of graphs g which "approximate" a decomposition. There are two approaches to this. We describe the two approaches in terms of the sample comparison analogy. In the first approach, we can try comparing as many of the samples as possible, without repetition of comparisons (it might be that running the machine is expensive). In the setting mentioned above, we could seek a collection of runs of the machine (represented by copies of K_3) which do not repeat pairs of samples run together (i.e., the copies of K_3 are edge disjoint), and which minimizes the number of pairs of samples which are omitted (i.e., the cardinality of the set of edges in K_v which are in none of the copies of K_3 is made minimal). Such an experimental design is related to a maximal graph packing. A *maximal g-packing* of a graph G with isomorphic copies of a graph g is a set $\{g_1, g_2, \dots, g_n\}$, where $g_i \cong g$ and $V(g_i) \subset V(G)$ for all i, $E(g_i) \cap E(g_j) = \varnothing$ for $i \neq j$, $\cup_{i=1}^{n} g_i \subset G$, and $|E(G) \setminus \cup_{i=1}^{n} E(g_i)|$ is minimal. In particular, the machine analogy corresponds to a K_3-packing of K_v. Such designs are explored in [2]. Other packings of the complete graphs have also been studied, for example, 4-cycle-packings [3], K_4-packings [4], and 6-cycle-packings [5, 6]. A second approach involves comparing *all* of the samples to each other, but with minimal repetitions of the compared samples (we might postulate that the machine must have three samples in it during each run to keep it balanced). This experimental design is related to a minimal

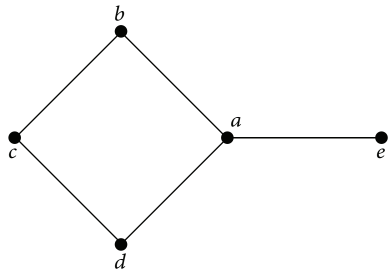

Figure 1: We denote by $H = [a, b, c, d; e]$ this graph.

graph covering. A *minimal g-covering* of a graph G with isomorphic copies of a graph g is a set $\{g_1, g_2, \dots, g_n\}$, where $g_i \cong g$, $V(g_i) \subset V(G)$, $E(g_i) \subset E(G)$ for all i, $G \subset \cup_{i=1}^{n} g_i$, and $|\cup_{i=1}^{n} E(g_i) \setminus E(G)|$ is minimal (when considering coverings, the graph $\cup_{i=1}^{n} g_i$ may not be simple and $\cup_{i=1}^{n} E(g_i)$ may be a multiset). The machine analogy in this case corresponds to a K_3-covering of K_v. Such designs are explored in [7]. Coverings of K_v have also been explored, for example, for 4-cycles [2] and 6-cycles [8].

In terms of graph decompositions, several studies have concentrated on the g-decompositions of complete graphs into copies of a given graph g with a small number of vertices [9–12]. In this paper, we go in a different direction and consider a single graph g, the 4-cycle with a pendant edge, and explore packings and coverings of several graphs related to the complete graph. We denote the 4-cycle with a pendant edge as $H = [a, b, c, d; e]$, where $V(H) = \{a, b, c, d, e\}$ and $E(H) = \{(a, b), (b, c), (c, d), (a, d), (a, e)\}$. See Figure 1. An H-decomposition of K_v exists if and only if $v \equiv 0$ or $1 \pmod 5$, $v \geq 10$ [9]. An H-decomposition of the complete bipartite graph, $K_{m,n}$, exists if and only if $mn \equiv 0 \pmod 5$, $m \geq 5$, and $n \geq 2$ [13]. Another graph related to the complete graph is the complete graph with a hole $K(v, w)$. The complete graph on v vertices with a hole of size w is the graph with a vertex set $V(K(v, w)) = V_{v-w} \cup V_w$, where $|V_{v-w}| = v - w$ and $|V_w| = w$, and edge set $E(K(v, w)) = \{(a, b) \mid a, b \in V(K(v, w)), \{a, b\} \not\subset V_w\}$. Necessary and sufficient conditions for the decomposition of $K(v, w)$ into m-cycles are known for $m \in \{3, 4, 5, 6, 7, 8, 10, 12, 14\}$ [14–16]. There is an H-decomposition of $K(v, w)$ if and only if $|E(K(v, w))| \equiv 0 \pmod 5$, $v - w \geq 4$, and $(v, w) \notin \{(5, 1), (6, 1)\}$ [13].

The graph $K(v, w)$ relates to the experimental design story as follows. Suppose you have performed comparisons on a collection of w samples and then received an additional collection of samples (say, $v - w$ new samples). You now wish to compare the $v - w$ new samples to each other and to the original w samples. In the case of the machine described above, this would correspond to a K_3 decomposition of $K(v, w)$. In the event that a decomposition does not exist, we can explore the packings and coverings of $K(v, w)$. With a maximal g-packing of G, we require that each copy of g is a subgraph of G. The definition given above for a maximal g-covering also involves the condition that each copy of g is a subgraph of G. Most studies of coverings have involved $G = K_v$, so the condition that the copies of g are subgraphs of G is trivially satisfied. But when G is not a complete graph, there is no obvious reason to impose the subgraph condition. Returning to the testing-of-samples story, we see no reason to disallow, for example, the testing (or retesting) of two samples in the hole of $K(v, w)$. Therefore, we are motivated to refine the definition of a graph covering into two cases—one case in which the edges that are not in G are forbidden from use in the copies of g and a second case in which these edges are not forbidden. A *minimal unrestricted g-covering* of a graph G with isomorphic copies of a graph g is a set $\{g_1, g_2, \dots, g_n\}$ where $g_i \cong g$, $V(g_i) \subset V(G)$, $G \subset \cup_{i=1}^{n} g_i$, and $|\cup_{i=1}^{n} E(g_i) \setminus G|$ is minimal (the graph $\cup_{i=1}^{n} g_i$ may not be simple and $\cup_{i=1}^{n} E(g_i)$ may be a multiset). A *minimal restricted g-covering* of a graph G with isomorphic copies of a graph g is a set $\{g_1, g_2, \dots, g_n\}$, where $g_i \cong g$, $V(g_i) \subset V(G)$, $E(g_i) \subset E(G)$ for all i, $G \subset \cup_{i=1}^{n} g_i$, and $|\cup_{i=1}^{n} E(g_i) \setminus G|$ is minimal. The distinction between restricted and unrestricted coverings was introduced in [17]. Notice that in the event that G is a complete graph, there is no distinction between a minimal restricted and minimal unrestricted covering.

The purpose of this paper is to give H-packings of K_v, $K_{m,n}$, and $K(v, w)$, as well as H-coverings of K_v, and restricted and unrestricted H-coverings of $K_{m,n}$ and $K(v, w)$.

2. Packing and Covering K_v

In this section, when necessary, we assume that the vertex set of K_v is $V(K_v) = \{0, 1, 2, \dots, v - 1\}$. Since H has 5 vertices, we only consider $v \geq 5$.

Theorem 1. *A maximal H-packing of K_v, $v \geq 5$, has leave L, where $|E(L)| = |E(K_v)| \pmod 5$, except when $v \in \{5, 6\}$ in which case $|E(L)| = 5$.*

Proof. Since $|E(H)| = 5$, then it is necessary that in any H-packing of K_v with leave L, $|E(L)| \equiv |E(K_v)| \pmod 5$. Therefore, such a packing with $|E(L)| = |E(K_v)| \pmod 5$ would be maximal. If $v \in \{5, 6\}$, then $|E(K_v)| \equiv 0 \pmod 5$, but there is not an H-decomposition of K_v [9]. So for $v \in \{5, 6\}$, an H-packing of K_v with leave L, where $|E(L)| = 5$ would be maximal.

Case 1. Suppose $v = 5$. The set $\{[0, 1, 2, 3; 4]\}$ is a maximal packing of K_5 with leave L, where $E(L) = \{(0, 2), (1, 3), (1, 4), (2, 4), (3, 4)\}$, so $|E(L)| = 5$.

Case 2. Suppose $v = 6$. The set $\{[0, 1, 2, 3; 4], [1, 3, 4, 5; 4]\}$ is a maximal packing of K_6 with leave L, where $E(L) = \{(0, 2), (0, 5), (2, 5), (2, 4), (3, 5)\}$, so $|E(L)| = 5$.

Case 3. Suppose $v \equiv 2$ or $4 \pmod 5$, $v \geq 9$. Since $|E(K_v)| \equiv 1 \pmod 5$, $|E(L)| = 1$ would be optimal. Now $K(v, 2)$ can be decomposed [13], so $|E(L)| = 1$.

Case 4. Suppose $v \equiv 3 \pmod 5$, $v \geq 8$. Since $|E(K_v)| \equiv 3 \pmod 5$, $|E(L)| = 3$ would be optimal. Now $K(v, 3)$ can be decomposed [13], so $|E(L)| = 3$. □

In the following result (and throughout this paper), we refer to an equality of the form $a = b \pmod c$. By this, we mean that $a \in \{0, 1, 2, \dots, c - 1\}$ and $a \equiv b \pmod c$.

Theorem 2. *A minimal H-covering of K_v, $v \geq 5$, has padding P, where $|E(P)| = -|E(K_v)| \pmod 5$, except when $v \in \{5, 6\}$ in which case $|E(P)| = 5$.*

Proof. Since $|E(H)| = 5$, then it is necessary that in any H-covering of K_v with padding P, we have $|E(K_n)| + |E(P)| \equiv 0 \pmod 5$ or that $|E(P)| \equiv -|E(K_v)| \pmod 5$. So if $|E(P)| = -|E(K_v)| \pmod 5$, then the covering is minimal. If $v \in \{5, 6\}$, then $-|E(K_v)| \equiv 0 \pmod 5$, but there is no H-decomposition of K_v [9]. So for $v \in \{5, 6\}$, an H-covering of K_v with padding P, where $|E(P)| = 5$ would be minimal.

Case 1. Suppose $v = 5$. The set $\{[0, 1, 2, 3; 4], [1, 2, 0, 4; 3], [4, 1, 0, 2; 3]\}$ is a minimal covering of K_5 with padding P, where $(P) = \{(0, 1), (0, 2), (0, 4), (1, 2), (1, 4)\}$, so $|E(P)| = 5$.

Case 2. Suppose $v = 6$. The set $\{[0, 1, 2, 3; 4], [5, 0, 2, 4; 1], [5, 3, 4, 1; 2], [3, 4, 5, 2; 1]\}$ is a minimal covering of K_6 with padding P, where $E(P) = \{(1, 5), (2, 3), (2, 5), (3, 4), (4, 5)\}$, so $|E(P)| = 5$.

Case 3. Suppose $v \equiv 2$ or $4 \pmod 5$, $v \geq 7$. There is an H-decomposition of $K(v, 2)$ [13]. Take such a decomposition, along with another copy of H which includes the edge of the hole of $K(v, 2)$. This gives a covering of K_v with padding P, where $|E(P)| = 4 = -|E(K_v)| \pmod 5$.

Case 4. Suppose $v \equiv 3 \pmod 5$, $v \geq 8$. An H-covering of K_8 is given by $\{[0, 1, 2, 7; 3], [1, 3, 5, 7; 6], [4, 5, 6, 3; 7], [2, 4, 6, 0; 5], [1, 4, 0, 5; 2], [7, 3, 2, 6; 0]\}$ with padding P, where $E(P) = \{(0,7), (1, 2)\}$ and the covering is optimal. For $v \geq 13$, $K_v = K(v, 8) \cup K_8$, $K(v, 8)$ can be decomposed [13], and K_8 can be covered with padding P, where $|E(P)| = 2$. Therefore, there is an optimal H-covering of K_v with padding P, where $|E(P)| = 2 = -|E(K_v)| \pmod 5$. □

3. Packing and Covering the Complete Bipartite Graph

In this section, we consider the H-packings and H-coverings of the complete bipartite graph $K_{m,n}$. We assume the partite sets of $K_{m,n}$ are $\{0_0, 1_0, \dots, (m-1)_0\}$ and $\{0_1, 1_1, \dots, (n-1)_1\}$.

Theorem 3. *A maximal H-packing of $K_{m,n}$ has leave L, where*

(1) *$|E(L)| = mn$ if $m = 1$ or $n = 1$, or if $m = n = 2$, or*

(2) *$|E(L)| = |E(K_{m,n})| \pmod 5$, otherwise.*

Proof. First, if m or n equals 1, then H is not a subgraph of $K_{m,n}$, and the leave must have mn edges. Similarly, the leave of a packing of $K_{2,2}$ has $mn = 4$ edges. For $m \geq 2$ and $n \geq 3$, as in the proof of Theorem 1, an H-packing of $K_{m,n}$ with leave L, where $|E(L)| = |E(K_{m,n})| \pmod 5$ would be maximal. Next, for $m \geq 2$ and $n \geq 3$ we observe that if there is a packing of $K_{m,n}$ with leave L, then there is a packing of $K_{m+5i,n+5j}$ with leave L for all $i, j \in \mathbb{N}$. This is because $K_{m+5i,n+5j} = K_{m,n} \cup K_{m,5j} \cup K_{5i,n} \cup K_{5i,5j}$, where the partite sets of $K_{m+5i,n+5j}$ are $\{0_0, 1_0, \dots, (m-1+5i)_0\}$ and $\{0_1, 1_1, \dots, (n-1+5j)_1\}$, the partite sets of $K_{m,n}$ are $\{0_0, 1_0, \dots, (m-1)_0\}$ and $\{0_1, 1_1, \dots, (n-1)_1\}$, the partite sets of $K_{m,5j}$ are $\{0_0, 1_0, \dots, (m-1)_0\}$ and $\{n_1, (n+1)_1, \dots, (n-1+5j)_1\}$, the partite sets of $K_{5i,n}$ are $\{m_0, (m+1)_0, \dots, (m-1+5i)_0\}$ and $\{0_1, 1_1, \dots, (n-1)_1\}$, and the partite sets of $K_{5i,5j}$ are $\{m_0, (m+1)_0, \dots, (m-1+5i)_0\}$ and $\{n_1, (n+1)_1, \dots, (n-1+5j)_1\}$. There is an H-decomposition of $K_{m,5j}$, $K_{5i,n}$, and $K_{5i,5j}$ [13].

In Table 1, the packings, combined with the decompositions of complete bipartite graphs mentioned above, yield the result. □

Theorem 4. *A minimal restricted H-covering of $K_{m,n}$, where neither m nor n equals 1 and $m+n \geq 5$, has padding P, where $|E(P)| = -|E(K_{m,n})| \pmod 5$.*

Proof. For $K_{1,n}$, H is not a subgraph and so a restricted H-covering does not exist. Similar to the argument in Theorem 2, a H-covering of $K_{m,n}$ with padding P where $|E(P)| = -|E(K_{m,n})| \pmod 5$ would be minimal. As in Theorem 3, for $m \geq 2$ and $m \geq 3$, if there is a restricted covering of $K_{m,n}$ with padding P, then there is a restricted covering of $K_{m+5i,n+5j}$ with padding P for all $i, j \in \mathbb{N}$.

In Table 2, the coverings, combined with the decompositions of complete graphs mentioned in Theorem 3, yield the result. □

Theorem 5. *A minimal unrestricted H-covering of $K_{m,n}$ has padding P where*

(1) *when $m > 1$ and $n > 1$, $|E(P)| = -|E(K_{m,n})| \pmod 5$,*

(2) *when $m = 1$, $|E(P)| = (2/3)n$ for $n \equiv 0 \pmod 3$, $|E(P)| = 2(n+5)/3$ for $n \equiv 1 \pmod 3$, $|E(P)| = (2n+5)/3$ for $n \equiv 2 \pmod 3$.*

Proof. For $m > 1$ and $n > 1$, the necessary condition follows as in the proof of Theorem 4. In this case, sufficiency also follows from Theorem 4.

When $m = 1$, a copy of H where $V(H) \subset V(K_{1,n})$ has at most 3 edges in $E(K_{1,n})$ and at least 2 edges in the padding. So in an H-covering of $K_{1,n}$, there are at least $\lceil n/3 \rceil$ copies of H. Now $\lfloor n/3 \rfloor$ copies of H can have at most $3\lfloor n/3 \rfloor$ edges in $E(K_{1,n})$ and at least $2\lfloor n/3 \rfloor$ edges in the padding. If $n \equiv 1 \pmod 3$, then to completely cover $K_{1,n}$ we must add one more copy of H which has at most 1 edge in $E(K_{1,n})$ and at least 4 edges in the padding. If $n \equiv 2 \pmod 3$, then to completely cover $K_{1,n}$ we must add one more copy of H which has at most 2 edges in $E(K_{1,n})$ and at least 3 edges in the padding. This yields the necessary conditions for $m = 1$. We now establish sufficiency for $m = 1$.

Case 1. Suppose $m = 1$ and $n \equiv 0 \pmod 3$; $n \geq 6$. Consider the blocks $\{[0_0, 0_1, 3_1, 1_1; 2_1]\} \cup \{[0_0, (3k)_1, 2_1, (3k+1)_1; (3k+2)_1] \mid k = 1, 2, \dots, (n/3) - 1\}$. This is a covering of $K_{m,n}$ with padding $P = \{(0_1, 3_1), (1_1, 3_1)\} \cup \{(2_1, (3k)_1), (2_1, (3k+1)_1) \mid k = 1, 2, \dots, (n/3) - 1\}$, where $|E(P)| = (2/3)n$.

Case 2. Suppose $m = 1$ and $n \equiv 1 \pmod 3$; $n \geq 4$. From Case 1, there is a covering of $K_{1,n-1}$, where the partite sets of $K_{1,n-1}$

Table 1

(m, n) (mod 5)	$K_{m,n}$	Packing	Leave
$(1, 1)$	$K_{6,6}$	$\{[0_0, 0_1, 1_0, 1_1; 2_1], [0_0, 5_1, 1_0, 4_1; 3_1], [2_0, 0_1, 3_0, 1_1; 2_1], [2_0, 4_1, 3_0, 5_1; 3_1], [4_0, 0_1, 5_0, 1_1; 2_1], [4_0, 4_1, 5_0, 5_1; 3_1], [2_1, 1_0, 3_1, 3_0; 5_0]\}$	$\{(5_0, 3_1)\}$
$(1, 2)$	$K_{6,2}$	$\{[0_1, 0_0, 1_1, 1_0; 2_0], [1_1, 5_0, 0_1, 4_0; 3_0]\}$	$\{(2_0, 1_1), (3_0, 0_1)\}$
$(1, 3)$	$K_{6,3}$	$\{[3_0, 0_1, 2_0, 1_1; 2_1], [1_0, 0_1, 0_0, 1_1; 2_1], [5_0, 0_1, 4_0, 1_1; 2_1]\}$	$\{(0_0, 2_1), (2_0, 2_1), (4_0, 2_1)\}$
$(1, 4)$	$K_{6,4}$	$\{[0_0, 0_1, 1_0, 1_1; 2_1], [3_1, 2_0, 2_1, 3_0; 1_0], [2_1, 4_0, 1_1, 5_0; 1_0], [3_1, 4_0, 0_1, 5_0; 0_0]\}$	$\{(2_0, 0_1), (2_0, 1_1), (3_0, 0_1), (3_0, 1_1)\}$
$(2, 2)$	$K_{2,2}$	$\varnothing$	$\{(0_0, 0_1), (0_0, 1_1), (1_0, 0_1), (1_0, 1_1)\}$
$(2, 3)$	$K_{3,2}$	$\{[1_1, 1_0, 0_1, 0_0; 2_0]\}$	$\{(2_0, 0_1)\}$
$(2, 4)$	$K_{4,2}$	$\{[1_1, 1_0, 0_1, 0_0; 2_0]\}$	$\{(2_0, 0_1), (3_0, 0_1), (3_0, 1_1)\}$
$(3, 3)$	$K_{3,3}$	$\{[1_1, 1_0, 0_1, 0_0; 2_0]\}$	$\{(0_0, 2_1), (1_0, 2_1), (2_0, 0_1), (2_0, 2_1)\}$
$(3, 4)$	$K_{4,3}$	$\{[0_0, 0_1, 1_0, 1_1; 2_1], [2_0, 3_1, 1_0, 2_1; 1_1]\}$	$\{(0_0, 3_1), (2_0, 0_1)\}$
$(4, 4)$	$K_{4,4}$	$\{[1_0, 1_1, 0_0, 0_1; 2_1], [3_0, 1_1, 2_0, 0_1; 2_1], [3_1, 2_0, 2_1, 0_0; 3_0]\}$	$\{(1_0, 3_1)\}$

Table 2

(m, n) (mod 5)	$K_{m,n}$	Covering	Padding
$(1, 1)$	$K_{6,6}$	$\{[3_1, 0_0, 0_1, 1_0; 5_0]\}$	$\{(0_0, 3_1), (0_0, 0_1), (1_0, 0_1), (1_0, 3_1)\}$
$(1, 2)$	$K_{6,2}$	$\{[0_1, 2_0, 1_1, 1_0; 3_0]\}$	$\{(1_0, 0_1), (1_0, 1_1), (2_0, 0_1)\}$
$(1, 3)$	$K_{6,3}$	$\{[2_1, 0_0, 1_1, 2_0; 4_0]\}$	$\{(0_0, 1_1), (2_0, 1_1)\}$
$(1, 4)$	$K_{6,4}$	$\{[2_0, 0_1, 3_0, 1_1; 2_1]\}$	$\{(2_0, 2_1)\}$
$(2, 2)$	$K_{7,2}$	$\{[0_1, 0_0, 1_1, 1_0; 6_0], [1_1, 2_0, 0_1, 3_0; 6_0], [1_1, 5_0, 0_1, 4_0; 0_0]\}$	$\{(0_0, 1_1)\}$
$(2, 3)$	$K_{3,2}$	$\{[0_1, 1_0, 1_1, 0_0; 2_0]\}$	$\{(0_0, 0_1), (0_0, 1_1), (1_0, 0_1), (1_0, 1_1)\}$
$(2, 4)$	$K_{4,2}$	$\{[0_1, 3_0, 1_1, 0_0; 2_0]\}$	$\{(0_0, 0_1), (0_0, 1_1)\}$
$(3, 3)$	$K_{3,3}$	$\{[2_1, 2_0, 0_1, 1_0; 0_0]\}$	$\{(1_0, 0_1)\}$
$(3, 4)$	$K_{4,3}$	$\{[0_1, 0_0, 3_1, 2_0; 1_0]\}$	$\{(0_0, 0_1), (1_0, 0_1), (2_0, 3_1)\}$
$(4, 4)$	$K_{4,4}$	$\{[3_1, 0_0, 0_1, 3_0; 1_0]\}$	$\{(0_0, 3_1), (0_0, 0_1), (3_0, 0_1), (3_0, 3_1)\}$

are $\{0_0\}$, and $V_n \setminus \{(n-1)_1\}$ with padding P_1, where $|E(P_1)| = 2(n-1)/3$. This covering along with $\{[0_0, 0_1, 2_1, 1_1; 3_1]\}$, is an unrestricted covering of $K_{m,n}$ with padding $P_2 = P_1 \cup \{(0_0, 0_1), (0_1, 2_1), (2_1, 1_1), (0_0, 1_1)\}$ and so $|E(P_2)| = 2(n+5)/3$.

Case 3. Suppose $m = 1$ and $n \equiv 2 \pmod 3$; $n \geq 5$. From Case 1, there is a covering of $K_{1,n-2}$, where the partite sets of $K_{1,n-2}$ are $\{0_0\}$ and $V_n \setminus \{(n-2)_1\}$ with padding P_1, where $|E(P_1)| = 2(n-2)/3$. This covering along with $\{[0_0, (n-1)_1, 0_1, (n-2)_1; 1_1]\}$ is an unrestricted covering of $K_{m,n}$ with padding $P_2 = P_1 \cup \{(0_1, (n-1)_1), (0_1, (n-2)_1), (0_0, 1_1)\}$, and so $|E(P_2)| = (2n+5)/3$. □

4. Packing the Complete Graph with a Hole

In this section, we assume the vertex set of $K(v, w)$ is $V(K(v, w)) = V_{v-w} \cup V_w$ as described in Section 1, where $V_{v-w} = \{0_0, 1_0, \ldots, (v-w-1)_0\}$ and $V_w = \{0_1, 1_1, \ldots, (w-1)_1\}$.

Theorem 6. *A maximal H-packing of $K(v, w)$ has leave L, where $|E(L)| = |E(K(v,w))| \pmod 5$ and $v - w \geq 2$ is necessary.*

Proof. When $v = w + 1$, H is not a subgraph of $K(v, w)$, and so, there is no packing. Therefore, $v - w \geq 2$ is necessary for the existence of a packing.

Case 1. If $v - w = 6$, then $K(v, w) = K_6 \cup K_{6,w}$, where the vertex set of K_6 is V_{v-w} and the partite sets of $K_{6,w}$ are V_{v-w} and V_w. There exists a packing $K_{6,w}$ with leave L_2 such that $|E(L_2)| \in \{1, 2, 3, 4\}$. Without loss of generality, $(0_0, 0_1) \in E(L_2)$. Take such a packing along with $\{[4_0, 5_0, 2_0, 3_0; 1_0], [0_0, 3_0, 1_0, 5_0; 2_0], [0_0, 1_0, 2_0, 4_0; 0_1]\}$. This yields a packing of $K(v, w)$ with leave $L = \{(3_0, 5_0)\} \cup E(L_2) \setminus \{(0_0, 0_1)\}$, so $|E(L)| = |E(L_2)|$.

Case 2. Suppose $v \equiv 0 \pmod 5$ and $w \equiv 2 \pmod 5$. Then $K(v, w) = K_{v-w} \cup K_{v-w,w}$, where $V(K_{v-w}) = V_{v-w}$ and the partite sets of $K_{v-w,w}$ are V_{v-w} and V_w. We have $v - w \equiv 3 \pmod 5$ and $w \equiv 2 \pmod 5$. There is a maximal packing of K_{v-w}, where $v - w \equiv 3 \pmod 5$ with $|E(L_1)| = 3$ by Theorem 1 and a maximal packing of $K_{v-w,w}$ with $|E(L_2)| = 1$ by Theorem 3. Therefore, there is a maximal packing of $K(v, w)$ with leave L, where $|E(L)| = 4 = |E(K(v, w))| \pmod 5$.

Case 3. Suppose $v \equiv 0 \pmod 5$ and $w \equiv 3 \pmod 5$ or $v \equiv 3 \pmod 5$ and $w \equiv 4 \pmod 5$. Then $K(v, w) = K_{v-w} \cup K_{v-w,w}$

as in Case 2, where $v - w \equiv 2 \pmod 5$ and $w \equiv 3 \pmod 5$ or $v - w \equiv 4 \pmod 5$ and $w \equiv 4 \pmod 5$. There is a maximal packing of K_{v-w}, where $v-w \equiv 2 \pmod 5$ or $v-w \equiv 4 \pmod 5$ with $|E(L_1)| = 1$ by Theorem 1, and there is a maximal packing of $K_{v-w,w}$ with $|E(L_2)| = 1$ by Theorem 3. Therefore, there is a maximal packing of $K(v,w)$ with leave L, where $|E(L)| = 2 = |E(K(v,w))| \pmod 5$.

Case 4. Suppose $v \equiv 0 \pmod 5$ and $w \equiv 4 \pmod 5$. When $v - w \geq 11$, $K(v,w) = K_{v-w} \cup K_{v-w,w}$, as in Case 2, where $v - w \equiv 1 \pmod 5$ and $w \equiv 4 \pmod 5$. There is a maximal packing of $K_{v-w,w}$ with $|E(L_2)| = 4$ by Theorem 3 and K_{v-w}, where $v-w \equiv 1 \pmod 5$ is decomposable [9]. Therefore, there is a maximal packing of $K(v,w)$ with leave L, where $|E(L)| = 4 = |E(K(v,w))| \pmod 5$.

Case 5. Suppose $v \equiv 1 \pmod 5$ and $w \equiv 2 \pmod 5$ or $v \equiv 1 \pmod 5$ and $w \equiv 4 \pmod 5$. Then $K(v,w) = K_{v-w} \cup K_{v-w,w}$, as in Case 2, where $v - w \equiv 4 \pmod 5$ and $w \equiv 2 \pmod 5$, or $v - w \equiv 2 \pmod 5$ and $w \equiv 4 \pmod 5$. There is a maximal packing of K_{v-w}, where $v-w \equiv 4 \pmod 5$ or $v-w \equiv 2 \pmod 5$ with $|E(L_1)| = 1$ by Theorem 1, and there is a maximal packing of $K_{v-w,w}$ with $|E(L_2)| = 3$ by Theorem 3. Therefore, there is a maximal packing of $K(v,w)$ with leave L, where $|E(L)| = 4 = |E(K(v,w))| \pmod 5$.

Case 6. Suppose $v \equiv 1 \pmod 5$ and $w \equiv 3 \pmod 5$. Then $K(v,w) = K_{v-w+1} \cup K_{v-w,w-1}$, where $V(K_{v-w+1}) = V_{v-w} \cup \{w_1\}$ and the partite sets of $K_{v-w,w-1}$ are $V_{v-w} \cup \{w_1\}$ and $V_w \setminus \{w_1\}$. Then there is a maximal packing of K_{v-w+1} with leave L_1, where $|E(L_1)| = 1$ by Theorem 1, and there is a maximal packing of $K_{v-w,w-1}$ with leave L_2, where $|E(L_2)| = 1$ by Theorem 3. Therefore, there is a maximal packing of $K(v,w)$ with leave L, where $|E(L)| = 2 = |E(K(v,w))| \pmod 5$.

Case 7. Suppose $v \equiv 2 \pmod 5$ and $w \equiv 0 \pmod 5$ or $v \equiv 4 \pmod 5$, and $w \equiv 0 \pmod 5$. Then $K(v,w) = K_{v-w} \cup K_{v-w,w}$ where $v - w \equiv 2 \pmod 5$ and $w \equiv 0 \pmod 5$ or $v - w \equiv 4 \pmod 5$, and $w \equiv 0 \pmod 5$. There is a maximal packing of K_{v-w}, where $v - w \equiv 2 \pmod 5$ or $v - w \equiv 4 \pmod 5$ with $|E(L_1)| = 1$ by Theorem 1 and $K_{v-w,w}$ is decomposable [13]. Therefore, there is a maximal packing of $K(v,w)$ with leave L where $|E(L)| = 1 = |E(K(v,w))| \pmod 5$.

Case 8. Suppose $v \equiv 2 \pmod 5$ and $w \equiv 1 \pmod 5$. Similar to Case 3, when $v-w \geq 11$, $K(v,w) = K_{v-w} \cup K_{v-w,w}$, as in Case 2, where $v - w \equiv 1 \pmod 5$ and $w \equiv 1 \pmod 5$. There is a maximal packing of $K_{v-w,w}$ with $|E(L_2)| = 1$ by Theorem 3 and K_{v-w}, where $v - w \equiv 1 \pmod 5$ is decomposable [9]. Therefore, there is a maximal packing of $K(v,w)$ with leave L, where $|E(L)| = 1 = |E(K(v,w))| \pmod 5$.

Case 9. Suppose $v \equiv 2 \pmod 5$ and $w \equiv 3 \pmod 5$, or $v \equiv 3 \pmod 5$ and $w \equiv 1 \pmod 5$. Then $K(v,w) = K_{v-w} \cup K_{v-w,w}$, as in Case 2, where $v-w \equiv 4 \pmod 5$, $w \equiv 3 \pmod 5$ or $v-w \equiv 2 \pmod 5$ and $w \equiv 1 \pmod 5$. There is a maximal packing of K_{v-w}, where $v - w \equiv 4 \pmod 5$ or $v - w \equiv 2 \pmod 5$ with $|E(L_1)| = 1$ by Theorem 1, and there is a maximal packing of $K_{v-w,w}$ with $|E(L_2)| = 2$ by Theorem 3. Therefore, there is a maximal packing of $K(v,w)$ with leave L, where $|E(L)| = 3 = |E(K(v,w))| \pmod 5$.

Case 10. Suppose $v \equiv 3 \pmod 5$ and $w \equiv 0 \pmod 5$. Then $K(v,w) = K_{v-w} \cup K_{v-w,w}$, as in Case 2, where $v - w \equiv 3 \pmod 5$ and $w \equiv 0 \pmod 5$. There is a maximal packing of K_{v-w}, where $v - w \equiv 3 \pmod 5$ with $|E(L_1)| = 3$ by Theorem 1 and $K_{v-w,w}$ is decomposable [13]. Therefore, there is a maximal packing of $K(v,w)$ with leave L, where $|E(L)| = 3 = |E(K(v,w))| \pmod 5$.

Case 11. Suppose $v \equiv 3 \pmod 5$ and $w \equiv 2 \pmod 5$. Similar to Case 4, when $v - w \geq 11$, $K(v,w) = K_{v-w} \cup K_{v-w,w}$, where $v - w \equiv 1 \pmod 5$ and $w \equiv 2 \pmod 5$. There is a maximal packing of $K_{v-w,w}$ with $|E(L_2)| = 2$ by Theorem 3 and K_{v-w} where $v-w \equiv 1 \pmod 5$ is decomposable [9]. Therefore, there is a maximal packing of $K(v,w)$ with leave L, where $|E(L)| = 2 = |E(K(v,w))| \pmod 5$.

Case 12. Suppose $v \equiv 4 \pmod 5$ and $w \equiv 1 \pmod 5$. As in Case 6, we have $K(v,w) = K_{v-w+1} \cup K_{v-w,w-1}$. Then there is a maximal packing of K_{v-w+1} with leave L_1, where $|E(L_1)| = 1$ by Theorem 1 and $K_{v-w,w-1}$ is decomposable [13]. There is a maximal packing of $K(v,w)$ with leave L, where $|E(L)| = 1 = |E(K(v,w))| \pmod 5$.

Case 13. Suppose $v \equiv 4 \pmod 5$ and $w \equiv 3 \pmod 5$. Similar to Case 4, when $v - w \geq 11$, consider $K(v,w) = K_{v-w} \cup K_{v-w,w}$, where $v - w \equiv 1 \pmod 5$ and $w \equiv 3 \pmod 5$. There is a maximal packing of $K_{v-w,w}$ with $|E(L_2)| = 3$ by Theorem 3 and K_{v-w}, where $v - w \equiv 1 \pmod 5$ is decomposable [9]. Therefore, there is a maximal packing of $K(v,w)$ with leave L, where $|E(L)| = 3 = |E(K(v,w))| \pmod 5$. □

5. Covering the Complete Graph with a Hole

As in the previous section, we assume the vertex set of $K(v,w)$ is $V(K(v,w)) = V_{v-w} \cup V_w$, where $V_{v-w} = \{0_0, 1_0, \dots, (v-w-1)_0\}$ and $V_w = \{0_1, 1_1, \dots, (w-1)_1\}$.

Theorem 7. *A minimal restricted H-covering of $K(v,w)$ has padding P, where $|E(P)| = -|E(K(v,w))| \pmod 5$, when $v - w > 2$. No restricted H-covering of $K(v,w)$ exists for $v - w = 2$.*

Proof. First, suppose $v - w = 2$. Consider the edge $(0_0, 1_0)$. If $(0_0, 1_0)$ is the pendant edge of an H, say $H = [0_0, a, b, c; 1_0]$, then $0_0, 1_0$, and b must be distinct vertices in V_{v-w}. But $|V_{v-w}| = 2$, so this cannot happen. If $(0_0, 1_0)$ is an edge in the 4-cycle of some H, then there must be an edge in the 4-cycle of the form (a_1, b_1), a contradiction to the restricted covering. So, $v - w > 2$ is necessary.

Similar to the argument in Theorem 2, an H-covering of $K(v,w)$ with padding P where $|E(P)| = -|E(K(v,w)| \pmod 5$ would be minimal.

Case 1. Suppose $v \equiv 0 \pmod 5$ and $w \equiv 2 \pmod 5$. First, $K(5,2)$ can be covered with $\{[0_0, 0_1, 2_0, 1_0; 1_1], [2_0, 0_1, 1_0, 1_1; 0_0]\}$, and this has a padding P with $E(P) = \{(2_0, 0_1)\}$ and so $|E(P)| = 1$. For general v and w, $K(v,w) = K(5,2) \cup K_{v-w-3,3} \cup$

$K_{v-w,w-2}$, where the vertex set of $K(5,2)$ is $\{0_0, 1_0, 2_0, 0_1, 1_1\}$ and the hole is on vertex set $\{0_1, 1_1\}$, the partite sets of $K_{v-w-3,3}$ are $\{3_0, 4_0, \ldots, (v-w-1)_0\}$ and $\{0_0, 1_0, 2_0\}$, and the partite sets of $K_{v-w,w-2}$ are V_{v-w} and $\{2_1, 3_1, \ldots, (w-1)_1\}$. Now, $K_{v-w-3,3}$ and $K_{v-w,w-2}$ can be decomposed [13]. Taking these decompositions along with the above covering of $K(5,2)$ yields a covering of $K(v,w)$ with padding P, where $E(P) = \{(2_0, 0_1)\}$, and so $|E(P)| = 1 = -|E(K(v,w))| \pmod 5$.

Case 2. Suppose $v \equiv 0 \pmod 5$ and $w \equiv 3 \pmod 5$ or $v \equiv 3 \pmod 5$, and $w \equiv 4 \pmod 5$. Consider $K(v,w) = K_{v-w} \cup K_{v-w,w}$, where $V(K_{v-w}) = V_{v-w}$ and the partite sets of $K_{v-w,w}$ are V_{v-w} and V_w and $v-w \equiv 2 \pmod 5$ and $w \equiv 3 \pmod 5$ or $v-w \equiv 4 \pmod 5$, and $w \equiv 4 \pmod 5$. There is a maximal packing of K_{v-w} where $v-w \equiv 2 \pmod 5$ or $v-w \equiv 4 \pmod 5$ with $|E(L_1)| = 1$ by Theorem 1. There is a maximal packing of $K_{v-w,w}$ with $|E(L_2)| = 1$ by Theorem 3. Therefore, there is a minimal covering of $K(v,w)$ with padding P, where $|E(P)| = 3 = -|E(K(v,w))| \pmod 5$.

Case 3. Suppose $v \equiv 0 \pmod 5$, and $w \equiv 4 \pmod 5$. Consider $K(v,w) = K_{v-w} \cup K_{v-w,w}$, as in Case 2, where $v-w \equiv 1 \pmod 5$, and $w \equiv 4 \pmod 5$. There is a minimal covering of $K_{v-w,w}$ with padding P, where $|E(P)| = 1$ by Theorem 4 and K_{v-w}, where $v-w \equiv 1 \pmod 5$ is decomposable [9]. Therefore, there is a minimal covering of $K(v,w)$ with padding P, where $|E(P)| = 1 = -|E(K(v,w))| \pmod 5$.

Case 4. Suppose $v \equiv 1 \pmod 5$ and $w \equiv 2 \pmod 5$. Consider $K(v,w) = K_{v-w} \cup K_{v-w,w}$, as in Case 2, where $v-w \equiv 4 \pmod 5$ and $w \equiv 2 \pmod 5$. There is a maximal packing of K_{v-w} with leave L_1, where $|E(L_1)| = 1$ by Theorem 1 and, without loss of generality, $E(L_1) = \{(0_0, 2_0)\}$. There is a maximal packing of $K_{v-w,w}$ with leave L_2, where $|E(L_2)| = 3$ and $E(L_2) = \{(2_0, 0_1), (0_1, 3_0), (3_0, 1_1)\}$ by Theorem 3. These two packings combined with $\{[2_0, 0_1, 3_0, 1_1; 0_0]\}$ yield a covering of $K(v,w)$ with padding P, where $E(P) = \{(2_0, 1_1)\}$, so $|E(P)| = 1 = -|E(K(v,w))| \pmod 5$.

Case 5. Suppose $v \equiv 1 \pmod 5$ and $w \equiv 4 \pmod 5$. Consider $K(v,w) = K_{v-w} \cup K_{v-w,w}$, as in Case 2, where $v-w \equiv 2 \pmod 5$ and $w \equiv 4 \pmod 5$. There is a maximal packing of K_{v-w} with leave L_1, where $|E(L_1)| = 1$ by Theorem 1 and, without loss of generality, $E(L_1) = \{(0_1, 2_1)\}$. There is a maximal packing of $K_{v-w,w}$ with leave L_2, where $|E(L_2)| = 3$ and $E(L_2) = \{(2_0, 0_1), (0_1, 3_0), (3_0, 1_1)\}$ by Theorem 3. These two packings combined with $\{[0_1, 3_0, 1_1, 2_0; 2_1]\}$ yield a covering of $K(v,w)$ with padding P, where $E(P) = \{(2_0, 1_1)\}$, so $|E(P)| = 1 = -|E(K(v,w))| \pmod 5$.

Case 6. Suppose $v \equiv 1 \pmod 5$ and $w \equiv 3 \pmod 5$. Consider $K(v,w) = K_{v-w+1} \cup K_{v-w,w-1}$, where $V(K_{v-w+1}) = V_{v-w} \cup \{w_1\}$, and the partite sets of $K_{v-w,w-1}$ are $V_{v-w} \cup \{w_1\}$ and $V_w \setminus \{w_1\}$. Then there is a maximal packing of K_{v-w+1} with leave L_1, where $|E(L_1)| = 1$ by Theorem 1, and there is a maximal packing of $K_{v-w,w-1}$ with leave L_2, where $|E(L_2)| = 1$ by Theorem 3. Therefore, we can add an additional copy of H which includes the edges in L_1 and L_2. So, there is a minimal covering of $K(v,w)$ with padding P, where $|E(P)| = 3 = -|E(K(v,w))| \pmod 5$.

Case 7. Suppose $v \equiv 2 \pmod 5$ and $w \equiv 0 \pmod 5$, or $v \equiv 4 \pmod 5$ and $w \equiv 0 \pmod 5$. Consider $K(v,w) = K_{v-w} \cup K_{v-w,w}$, as in Case 2, where $v-w \equiv 2 \pmod 5$, and $w \equiv 0 \pmod 5$ or $v-w \equiv 4 \pmod 5$, and $w \equiv 0 \pmod 5$. There is a minimal covering of K_{v-w} with padding P, where $|E(P)| = 4$ by Theorem 2, and $K_{v-w,w}$ is decomposable [13]. Therefore, there is a minimal covering of $K(v,w)$ with padding P where $|E(P)| = 4 = -|E(K(v,w))| \pmod 5$.

Case 8. Suppose $v \equiv 2 \pmod 5$ and $w \equiv 1 \pmod 5$. Consider $K(v,w) = K_{v-w} \cup K_{v-w,w}$, as in Case 2, where $v-w \equiv 1 \pmod 5$ and $w \equiv 1 \pmod 5$. There is a minimal covering of $K_{v-w,w}$ with padding P, where $|E(P)| = 4$ by Theorem 4 and K_{v-w}, where $v-w \equiv 1 \pmod 5$ is decomposable [9]. Therefore, there is a minimal covering of $K(v,w)$ with padding P, where $|E(P)| = 4 = -|E(K(v,w))| \pmod 5$.

Case 9. Suppose $v \equiv 2 \pmod 5$ and $w \equiv 3 \pmod 5$. Consider $K(v,w) = K_{v-w} \cup K_{v-w,w}$, as in Case 2, where $v-w \equiv 4 \pmod 5$ and $w \equiv 3 \pmod 5$. There is a maximal packing of K_{v-w} with leave L_1, where $|E(L_1)| = 1$ by Theorem 1 and, without loss of generality, $E(L_1) = \{(0_0, 1_0)\}$. There is a maximal packing of $K_{v-w,w}$ with leave L_2, where $|E(L_2)| = 2$ and $E(L_2) = \{(0_0, 3_1), (2_0, 0_1)\}$ by Theorem 3. These two packings combined with $\{[0_0, 0_1, 2_0, 3_1; 1_0]\}$ yield a covering of $K(v,w)$ with padding P, where $E(P) = \{(0_0, 0_1), (2_0, 3_1)\}$, so $|E(P)| = 2 = -|E(K(v,w))| \pmod 5$.

Case 10. Suppose $v \equiv 3 \pmod 5$ and $w \equiv 1 \pmod 5$. Consider $K(v,w) = K_{v-w} \cup K_{v-w,w}$, as in Case 2, where $v-w \equiv 2 \pmod 5$ and $w \equiv 1 \pmod 5$. There is a maximal packing of K_{v-w} with leave L_1, where $|E(L_1)| = 1$ by Theorem 1 and, without loss of generality, $E(L_1) = \{(1_0, 2_0)\}$. These is a maximal packing of $K_{v-w,w}$ with leave L_2, where $|E(L_2)| = 2$ and $E(L_2) = \{(2_0, 1_1), (3_0, 0_1)\}$ by Theorem 3. There two packings combined with $\{[2_0, 0_1, 3_0, 1_1; 1_0]\}$ yield a covering of $K(v,w)$ with padding P, where $E(P) = \{(2_0, 0_1), (3_0, 1_1)\}$, so $|E(P)| = 2 = -|E(K(v,w))| \pmod 5$.

Case 11. Suppose $v \equiv 3 \pmod 5$ and $w \equiv 0 \pmod 5$. Consider $K(v,w) = K_{v-w} \cup K_{v-w,w}$, as in Case 2, where $v-w \equiv 3 \pmod 5$ and $w \equiv 0 \pmod 5$. There is a minimal covering of K_{v-w} with padding P, where $|E(P)| = 2$ by Theorem 2, and $K_{v-w,w}$ is decomposable [13]. Therefore, there is a minimal covering of $K(v,w)$ with padding P, where $|E(P)| = 2 = -|E(K(v,w))| \pmod 5$.

Case 12. Suppose $v \equiv 3 \pmod 5$ and $w \equiv 2 \pmod 5$. Consider $K(v,w) = K_{v-w} \cup K_{v-w,w}$, as in Case 2, where $v-w \equiv 1 \pmod 5$ and $w \equiv 2 \pmod 5$. There is a minimal covering of $K_{v-w,w}$ with padding P, where $|E(P)| = 3$ by Theorem 4 and K_{v-w}, where $v-w \equiv 1 \pmod 5$ is decomposable [9]. Therefore, there is a minimal covering of $K(v,w)$ with padding P, where $|E(P)| = 3 = -|E(K(v,w))| \pmod 5$.

Case 13. Suppose $v \equiv 4 \pmod 5$ and $w \equiv 1 \pmod 5$. As in Case 6, we have $K(v,w) = K_{v-w+1} \cup K_{v-w,w-1}$. Then there is a minimal covering of K_{v-w+1} with padding P, where $|E(P)| = 4$ by Theorem 2 and $K_{v-w,w-1}$ is decomposable [13]. Therefore, there is a minimal covering of $K(v,w)$ with padding P, where $|E(P)| = 4 = -|E(K(v,w))| \pmod 5$.

Case 14. Suppose $v \equiv 4 \pmod 5$ and $w \equiv 3 \pmod 5$. Consider $K(v,w) = K_{v-w} \cup K_{v-w,w}$, as in Case 2, where $v - w \equiv 1 \pmod 5$ and $w \equiv 3 \pmod 5$. There is a minimal covering of $K_{v-w,w}$ with padding P, where $|E(P)| = 2$ by Theorem 4 and K_{v-w}, where $v-w \equiv 1 \pmod 5$ is decomposable [9]. Therefore, there is a minimal covering of $K(v,w)$ with padding P, where $|E(P)| = 2 = -|E(K(v,w))| \pmod 5$. □

Theorem 8. *A minimal unrestricted H-covering of $K(v,w)$ has padding P where*

(1) *when* $v - w > 2$, $|E(P)| = -|E(K(v,w))| (\mathrm{mod}\ 5)$,

(2) *when* $v - w = 1$, $|E(P)| = (2/3)w$ *for* $w \equiv 0 (\mathrm{mod}\ 3)$, $|E(P)| = 2(w+5)/3$ *for* $w \equiv 1 (\mathrm{mod}\ 3)$, $|E(P)| = (2w+5)/3$ *for* $w \equiv 2 (\mathrm{mod}\ 3)$,

(3) *when* $v - w = 2$, $|E(P)| = 5 - \ell$, *where* $\ell = |E(K(v,w)| (\mathrm{mod}\ 5)$ *for* $v \neq 6$ *and* $|E(P)| = 6$ *for* $v = 6$.

Proof. When $v - w > 2$, the necessary and sufficient conditions follow from Theorem 7. When $v-w = 1$, $K(v,w) \cong K_{1,w}$ and the necessary and sufficient conditions follow from Theorem 5.

When $v-w = 2$, similar to the argument in Theorem 2, an H-covering of $K(v,w)$ with padding P must satisfy $|E(P)| \equiv -|E(K(v,w)| \pmod 5$. Since an H-decomposition of $K(v,w)$ does not exist for $w \equiv 2 \pmod 5$ [13], the necessary conditions follow for $v - w = 2$ and $v \neq 6$. For $v = 6$, since $|E(K(6,4))| = 9$, then an unrestricted H-covering of $K(6,4)$ with padding P where $|E(P)| = 1$ would be minimal. However, in such a covering, there are only two copies of H. Edge $(0_0, 1_0)$ cannot be the pendant edge of a copy of H in such a covering since this copy would have 2 edges in the padding. If edge $(0_0, 1_0)$ is in a copy of H and is not the pendant edge, then this copy of H must be of the form $[0_0, 1_0, a_1, b_1; c_1]$ for some distinct $a_1, b_1, c_1 \in \{0_1, 1_1, 2_1, 3_1\}$. However, the complement of this graph in $K(6,4)$ is not a copy of H. Therefore, no such H-covering of $K(6,4)$ exists, and a minimal unrestricted H-covering of $K(6,4)$ with padding P, where $|E(P)| = 6$ would be minimal. The set $\{[1_0, 1_1, 0_1, 0_0; 2_1], [0_0, 2_1, 1_0, 3_1; 1_1], [1_0, 3_1, 0_0, 2_1; 0_1]\}$ is an unrestricted H-covering of $K(6,4)$ with padding P where $E(P) = \{(0_1, 1_1), (1_0, 2_1), (0_0, 2_1), (1_0, 2_1), (1_0, 3_1), (0_0, 3_1)\}$. So $|E(P)| = 6$, and the covering is minimal.

Case 1. Suppose $v - w = 2$ and $w \equiv 0 \pmod 5$; $w \geq 5$. Then $K(v,w) = K(7,5) \cup K_{2,w-5}$ where the vertex set of $K(7,5)$ is $\{0_0, 1_0, 0_1, 1_1, 2_1, 3_1, 4_1\}$ and the hole is on vertex set $\{0_1, 1_1, 2_1, 3_1, 4_1\}$, and the partite sets of $K_{2,w-5}$ are $\{0_0, 1_0\}$ and $\{5_1, 6_1, \ldots, (w-1)_1\}$. There is an H-decomposition of $K_{2,w-5}$ [13], and the set $\{[1_0, 1_1, 0_1, 0_0; 2_1], [0_0, 3_1, 1_0, 4_1; 2_1], [1_1, 0_1, 1_0, 2_1; 0_0]\}$ is an unrestricted H-covering of $K(7,5)$ with padding P, where $E(P) = \{(0_1, 1_1), (0_1, 1_1), (1_1, 2_1), (1_0, 2_1)\}$ and $|E(P)| = 4$. So, there is an unrestricted covering of $K(v,w)$ with padding P, where $|E(P)| = 4 = -|E(K(v,w))| \pmod 5$.

Case 2. Suppose $v - w = 2$, $w \equiv 1 \pmod 5$; $w \geq 6$. Then, as in Case 1, $K(v,w) = K(8,2) \cup K_{2,w-6}$. There is an H-decomposition of $K_{2,w-6}$ [13], and the set $\{[1_0, 1_1, 0_1, 0_0; 2_1], [1_0, 3_1, 0_0, 4_1; 0_1], [0_1, 2_1, 1_0, 5_1; 1_1]\}$ is an unrestricted H-covering of $K(8,6)$ with padding P, where $E(P) = \{(0_1, 1_1), (1_0, 2_1)\}$ and $|E(P)| = 2$. So, there is an unrestricted covering of $K(v,w)$ with padding P, where $|E(P)| = 2 = -|E(K(v,w))| \pmod 5$.

Case 3. Suppose $v - w = 2$ and $w \equiv 2 \pmod 5$; $w \geq 7$. Then, as in Case 1, $K(v,w) = K(9,7) \cup K_{2,w-7}$. There is an H-decomposition of $K_{2,w-7}$ [13], and the set $\{[1_0, 1_1, 0_1, 0_0; 2_1], [1_0, 3_1, 0_0, 4_1; 0_1], [0_1, 2_1, 1_0, 5_1; 1_1], [0_0, 5_1, 1_0, 6_1; 0_1]\}$ is an unrestricted H-covering of $K(9,7)$ with padding P, where $E(P) = \{(0_1, 1_1), (1_0, 2_1), (0_0, 0_1), (0_0, 5_1), (1_0, 5_1)\}$ and $|E(P)| = 5$. So, there is an unrestricted covering of $K(v,w)$ with padding P, where $|E(P)| = 5 = 5 - \ell$, where $\ell = 0 = -|E(K(v,w))| \pmod 5$.

Case 4. Suppose $v-w = 2$ and $w \equiv 3 \pmod 5$. Then, as in Case 1, $K(v,w) = K(5,3) \cup K_{2,w-3}$. There is an H-decomposition of $K_{2,w-3}$ [13], and the set $\{[1_0, 0_0, 0_1, 1_1; 2_1], [0_1, 2_1, 0_0, 1_1; 1_0]\}$ is an unrestricted H-covering of $K(5,3)$ with padding P, where $E(P) = \{(0_1, 1_1), (0_1, 1_1), (0_1, 2_1)\}$ and $|E(P)| = 3$. So, there is an unrestricted covering of $K(v,w)$ with padding Pn where $|E(P)| = 3 = -|E(K(v,w))| \pmod 5$.

Case 5. Suppose $v - w = 2$ and $w \equiv 4 \pmod 5$; $w \geq 9$. Then, as in Case 1, $K(v,w) = K(11,9) \cup K_{2,w-9}$. There is an H-decomposition of $K_{2,w-9}$ [13], and the set $\{[1_0, 1_1, 0_1, 0_0; 2_1], [0_0, 7_1, 1_0, 8_1; 1_1], [0_0, 5_1, 1_0, 6_1; 2_1], [0_0, 3_1, 1_0, 4_1; 0_1]\}$ is an unrestricted covering of $K(11,9)$ with padding P, where $E(P) = \{(0_1, 1_1)\}$ and $|E(P)| = 1$. So, there is an unrestricted covering of $K(v,w)$ with padding P, where $|E(P)| = 1 = -|E(K(v,w))| \pmod 5$. □

6. Conclusion

Motivated by experimental designs and comparisons of samples, we have given necessary and sufficient conditions for the H-packings and H-coverings of complete graphs, complete bipartite graphs, and complete graphs with a hole, where H is a 4-cycle with a pendant edge. For complete bipartite graphs and complete graphs with a hole, we have given both restricted and unrestricted coverings.

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