Bicyclic Steiner triple systems

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Abstract

A Steiner triple system admitting an automorphism whose disjoint cyclic decomposition consists of two cycles is said to be bicyclic. Necessary and sufficient conditions are given for the existence of bicyclic Steiner triple systems.

1. Introduction

A Steiner triple system of order \( v \), denoted STS or STS(\( v \)), is a \( v \)-element set \( X \) of points, together with a set \( \beta \), of unordered triples of elements of \( X \), called blocks, such that any two points of \( X \) are together in exactly one block of \( \beta \). It is well known that a STS(\( v \)) exists if and only if \( v \equiv 1 \) or \( 3 \) (mod 6). An automorphism of a STS is a permutation \( \pi \) of \( X \) which fixes \( \beta \). A permutation \( \pi \) of a \( v \)-element set is said to be of type \( [\pi]=[p_1, p_2, \ldots, p_v] \) if the disjoint cyclic decomposition of \( \pi \) contains \( p_i \) cycles of length \( i \). The orbit of a block under an automorphism, \( \pi \), is the image of the block under the powers of \( \pi \). A set of blocks \( B \) is said to be a set of base blocks for a STS under the permutation \( \pi \) if the orbits of the blocks of \( B \) produce the STS and exactly one block of \( B \) occurs in each orbit.

Several types of automorphisms have been explored in connection with the question of which orders \( v \) does there exist a STS(\( v \)) admitting an automorphism of the given type? A cyclic STS(\( v \)) is one admitting an automorphism of type \([0, 0, \ldots, 1]\) and exists if and only if \( v \equiv 1 \) or \( 3 \) (mod 6) and \( v \neq 9 \) \([5, 6, 7, 10]\). A reverse STS(\( v \)) admits an automorphism of type \([1, (v - 1)/2, 0, \ldots, 0]\). Reverse STS(\( v \))s exist if and only if \( v \equiv 1, 3, 9, \) or \( 19 \) (mod 24) \([3, 9, 11, 12]\). A \( k \)-rotational STS(\( v \)) admits an automorphism of type \([1, 0, 0, \ldots, 0, k, 0, \ldots, 0]\). \( k \)-rotational STSs have been addressed for \( k=1, 2, 3, 4, \) and \( 6 \) \([2, 8]\).

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In this paper, we explore \( STS(v) \)s admitting an automorphism of type \( [\pi] = [0, 0, \ldots, 0, p_{N_1}, 0, \ldots, 0, p_{N_2}, 0, \ldots, 0] \) where \( p_{N_1} = p_{N_2} = 1, N_1 < N_2 \) and \( N_1 + N_2 = v \). That is, the disjoint cyclic decomposition of \( \pi \) consists of one cycle of length \( N_1 \) and another (larger) cycle of length \( N_2 \). We call such systems bicyclic Steiner triple systems.

2. Previous results and necessary conditions

A bicyclic \( STS(v) \) with the smaller cycle of length \( 1 \) is also a \( 1 \)-rotational \( STS(v) \) and exists if and only if \( v \equiv 3 \) or \( 9 \pmod{24} \) [8]. So, henceforth, we will assume \( N_1 > 1 \). If \( N_1 = 3 \), then a bicyclic \( STS(v) \) admitting an automorphism of type \( [0, 0, 1, 0, \ldots, 0, 1, 0, 0, 0] \) exists if and only if \( v \equiv 3 \pmod{6} \) [1]. If \( N_1 = 7 \), then a bicyclic \( STS(v) \) admitting the relevant type of automorphism exists if and only if \( v \equiv 21 \pmod{42} \) [4]. Some necessary conditions, in particular the following lemmas, for the existence of bicyclic \( STS(v) \)s were given in [4].

**Lemma 2.1.** A bicyclic \( STS(v) \) admitting the above automorphism \( \pi \) satisfies the condition \( N_1 \equiv 1 \) or \( 3 \pmod{6} \), \( N_1 \neq 9 \), and \( N_1 | N_2 \).

**Lemma 2.2.** A bicyclic \( STS(v) \) admitting the above automorphism \( \pi \) with \( N_1 \equiv 1 \pmod{6} \) satisfies the condition, \( N_2 \equiv 2 \pmod{6} \). If \( N_1 \equiv 3 \pmod{6} \) then \( N_2 \equiv 0 \pmod{6} \) is necessary.

We will show that these necessary conditions are sufficient for \( N_1 > 1 \).

In our constructions, we will require the use of two structures. An \((A, n)\)-system is a collection of ordered pairs \((a_r, b_r)\) for \( r = 1, 2, \ldots, n \) that partition the set \( \{1, 2, \ldots, 2n\} \) with the property that \( b_r = a_r + r \) for \( r = 1, 2, \ldots, n \). An \((A, n)\)-system exists if and only if \( n \equiv 0 \) or \( 1 \pmod{4} \) [10]. A \((B, n)\)-system is a collection of ordered pairs \((a_r, b_r)\) for \( r = 1, 2, \ldots, n \) that partition the set \( \{1, 2, \ldots, 2n-1, 2n+1\} \) with the property that \( b_r = a_r + r \) for \( r = 1, 2, \ldots, n \). These systems exist if and only if \( n \equiv 2 \) or \( 3 \pmod{4} \) [6].

3. Constructions and sufficient conditions

In this section, we present several constructions to show that the necessary conditions of Lemmas 2.1 and 2.2 are sufficient. We will construct bicyclic \( STS(v) \)s on the set \( X = Z^*_N \times \{1\} \cup Z^*_N \times \{2\} \) with the automorphism being \( \pi = (0_1, 1_1, \ldots, (N_1 - 1)_1) (0_2, 1_2, \ldots, (N_2 - 1)_2) \). Since \( N_1 | N_2 \), we will let \( N_2 = kN_1 \) and express base blocks in terms of \( k \) and \( N_1 \).

**Lemma 3.1.** A bicyclic \( STS(v) \) on the set \( X \) admitting the automorphism \( \pi \) exists if:
Bicyclic Steiner triple systems

\[ N_1 \equiv 1 \pmod{24} \text{ and } k \equiv 2 \pmod{24}, \text{ or } \]
\[ N_1 \equiv 1 \pmod{24} \text{ and } k \equiv 8 \pmod{24}, \text{ or } \]
\[ N_1 \equiv 13 \pmod{24} \text{ and } k \equiv 14 \pmod{24}, \text{ or } \]
\[ N_1 \equiv 13 \pmod{24} \text{ and } k \equiv 20 \pmod{24}. \]

**Proof.** Under any one of these conditions,

\[ 3M = \frac{N_2}{2} - \frac{N_1 - 1}{2} - 1 \equiv 0 \text{ or } 3 \pmod{12} \]

and \( M \equiv 0 \text{ or } 1 \pmod{4} \). Consider the following collection of blocks:

\[ \left(0_1, \left(\frac{N_1 - 1}{4} + r\right), \left(\frac{(2k + 1)N_1 - 5}{4} - r\right)\right) \text{ for } r = 0, 1, \ldots, \frac{N_1 - 5}{4}, \]

\[ \left(0_1, \left(\frac{3N_1 + 1}{4} + r\right), \left(\frac{(2k + 3)N_1 - 7}{4} - r\right)\right) \text{ for } r = 0, 1, \ldots, \frac{N_1 - 5}{4}, \]

\[ \left(0_1, \left(\frac{3N_1 - 3}{4}\right), \left(\frac{(2k + 3)N_1 - 3}{4}\right)\right), \]

and \((0_2, r_2, (b_2 + M)_2)\) for \( r = 1, 2, \ldots, M \), where the \( a_r \) and \( b_r \) are from an \((A, M)\)-system.

This collection of blocks along with the base blocks for a cyclic \( \text{STS}(N_1) \) on \( \mathbb{Z}_{N_1} \times \{1\} \) under the automorphism \((0_1, 1, \ldots, (N_1 - 1)_1)\) form a complete set of base blocks for a bicyclic \( \text{STS}(v) \) with \( v = N_1 + N_2 \). \( \square \)

**Lemma 3.2.** A bicyclic \( \text{STS}(v) \) on the set \( X \) admitting the automorphism \( \pi \) exists if:

\[ N_1 \equiv 7 \pmod{24} \text{ and } k \equiv 2 \pmod{24}, \text{ or } \]
\[ N_1 \equiv 7 \pmod{24} \text{ and } k \equiv 8 \pmod{24}, \text{ or } \]
\[ N_1 \equiv 19 \pmod{24} \text{ and } k \equiv 14 \pmod{24}, \text{ or } \]
\[ N_1 \equiv 19 \pmod{24} \text{ and } k \equiv 20 \pmod{24}. \]

**Proof.** Under any one of these conditions,

\[ 3M = \frac{N_2}{2} - \frac{N_1 - 1}{2} - 1 \equiv 0 \text{ or } 3 \pmod{12} \]
and $M \equiv 0$ or $1 \pmod{4}$. Consider the following collection of blocks:

\[
\left(0, \left(\frac{N_1 + 1}{4} + r\right)^2, \left(\frac{(2k + 1)N_1 - 7}{4} - r\right)^2\right) \quad \text{for } r = 0, 1, \ldots, \frac{N_1 - 7}{4},
\]

\[
\left(0, \left(\frac{3N_1 - 1}{4} + r\right)^2, \left(\frac{(2k + 3)N_1 - 5}{4} - r\right)^2\right) \quad \text{for } r = 0, 1, \ldots, \frac{N_1 - 3}{4},
\]

\[
\left(0, \left(\frac{N_1 - 3}{4} + r\right)^2, \left(\frac{(2k + 1)N_1 - 3}{4} - r\right)^2\right),
\]

and $(0_2, r_2, (b_r + M) r_2)$ for $r = 1, 2, \ldots, M$, where the $a_r$ and $b_r$ are from an $(A,M)$-system.

This collection of blocks along with the base blocks for a cyclic $\text{STS}(N_1)$ on $\mathbb{Z}_N \times \{1\}$ under the automorphism $(0_1, 1_1, \ldots, (N_1 - 1)_1)$ form a complete set of base blocks for a bicyclic $\text{STS}(v)$ with $v = N_1 + N_2$. □

**Lemma 3.3.** A bicyclic $\text{STS}(v)$ on the set $X$ admitting the automorphism $\pi$ exists if:

- $N_1 \equiv 1 \pmod{24}$ and $k \equiv 14 \pmod{24}$, or
- $N_1 \equiv 1 \pmod{24}$ and $k \equiv 20 \pmod{24}$, or
- $N_1 \equiv 13 \pmod{24}$ and $k \equiv 2 \pmod{24}$, or
- $N_1 \equiv 13 \pmod{24}$ and $k \equiv 8 \pmod{24}$.

**Proof.** Under any one of these conditions,

\[3M = \frac{N_2}{2} - \frac{N_1 - 1}{2} - 1 \equiv 0 \text{ or } 3 \pmod{12}\]

and $M \equiv 2$ or $3 \pmod{4}$. Consider the following collection of blocks:

\[
\left(0, \left(\frac{N_1 - 1}{4} + r\right)^2, \left(\frac{(2k + 1)N_1 - 5}{4} - r\right)^2\right) \quad \text{for } r = 0, 1, \ldots, \frac{N_1 - 5}{4},
\]

\[
\left(0, \left(\frac{3N_1 + 1}{4} + r\right)^2, \left(\frac{(2k + 3)N_1 - 7}{4} - r\right)^2\right) \quad \text{for } r = 0, 1, \ldots, \frac{N_1 - 9}{4},
\]

\[
\left(0, \left(\frac{N_1 - 1}{2} + r\right)^2, \left(\frac{kN_1 - 1}{2} - 1\right)^2\right), \left(0, \left(\frac{3N_1 - 3}{4} + r\right)^2, \left(\frac{(2k + 3) - 3}{4} - r\right)^2\right),
\]

and $(0_2, r_2, (b_r + M) r_2)$ for $r = 1, 2, \ldots, M$, where the $a_r$ and $b_r$ are from a $(B,M)$-system.

This collection of blocks along with the base blocks for a cyclic $\text{STS}(N_1)$ on $\mathbb{Z}_N \times \{1\}$ under the automorphism $(0_1, 1_1, \ldots, (N_1 - 1)_1)$ form a complete set of base blocks for a bicyclic $\text{STS}(v)$ with $v = N_1 + N_2$. □
Lemma 3.4. A bicyclic STS(v) on the set X admitting the automorphism π exists if:

\[ N_1 \equiv 7 \pmod{24} \text{ and } k \equiv 14 \pmod{24}, \text{ or} \]
\[ N_1 \equiv 7 \pmod{24} \text{ and } k \equiv 20 \pmod{24}, \text{ or} \]
\[ N_1 \equiv 19 \pmod{24} \text{ and } k \equiv 2 \pmod{24}, \text{ or} \]
\[ N_1 \equiv 19 \pmod{24} \text{ and } k \equiv 8 \pmod{24}. \]

Proof. Under any one of these conditions,

\[ 3M = \frac{N_2}{2} - \frac{N_1 - 1}{2} - 1 \equiv 0 \text{ or } 3 \pmod{12} \]

and \( M \equiv 2 \pmod{3} \). Consider the following collection of blocks:

\[ \left( 0_1, \left( \frac{N_1 + 1}{4} + r \right)_2, \left( \frac{(2k+1)N_1 - 7}{4} - r \right)_2 \right) \text{ for } r = 0, 1, \ldots, \frac{N_1 - 7}{4}, \]
\[ \left( 0_1, \left( \frac{3N_1 - 1}{4} + r \right)_2, \left( \frac{(2k+3)N_1 - 5}{4} - r \right)_2 \right) \text{ for } r = 0, 1, \ldots, \frac{N_1 - 7}{4}, \]
\[ \left( 0_1, \frac{N_1 - 3}{4} _2, \left( \frac{(2k+1)N_1 - 3}{4} \right)_2, \frac{N_1 - 1}{2} _2, \left( \frac{kN_1 - 1}{2} - 1 \right)_2 \right) \]

and \( (0_2, r_2, (b_r + M)_2) \) for \( r = 1, 2, \ldots, M \), where the \( a_r \) and \( b_r \) are from a \((B,M)\)-system.

This collection of blocks along with the base blocks for a cyclic STS(\(N_1\)) on \(Z_{N_1} \times \{1\}\) under the automorphism \((0_1, 1_1, \ldots, (N_1 - 1)_1)\) form a complete set of base blocks for a bicyclic STS(v) with \( v = N_1 + N_2 \). \( \square \)

Lemmas 3.1–3.4 combine to give us the following theorem.

Theorem 3.5. A bicyclic STS(v) where \( v = N_1 + N_2 \) and \( N_2 = kN_1 \) admitting an automorphism whose disjoint cyclic decomposition is a cycle of length \( N_1 \) and a cycle of length \( N_2 \) exists if \( N_1 \equiv 1 \pmod{6}, N_1 > 1, \) and \( k \equiv 2 \pmod{6} \).

We now turn our attention to the case when \( N_1 \equiv 3 \pmod{6} \).

Lemma 3.6. A bicyclic STS(v) on the set X admitting the automorphism π exists if \( N_1 \equiv 3 \pmod{12}, N_1 > 3, \) and \( k \equiv 0 \pmod{12} \).

Proof. Consider the following collection of base blocks:

\[ \left( 0_2, \left( \frac{kN_1 - 24}{6} - 2r \right)_2, \left( \frac{kN_1 - 3}{3} - r \right)_2 \right) \text{ for } r = 0, 1, \ldots, \frac{kN_1 - 36}{12}, \]
\[
\left( 0, \frac{kN_1 - 18}{6} \right), \left( \frac{kN_1 - 2}{2} \right)
\]
for \( r = \frac{N_1 - 9}{6}, \frac{N_1 - 3}{6}, \ldots, \frac{kN_1 - 24}{12} \),

\[
\left( 0, \frac{kN_1 + 6}{6} \right), \left( \frac{kN_1 + 3}{3} \right), \left( 0, \frac{kN_1}{3} \right), \left( \frac{2kN_1}{3} \right)
\]

\[
\left( 0, \frac{N_1 - 3}{6} + r \right), \left( \frac{(k + 1)N_1 - 21}{6} \right)
\]
for \( r = 0, 1, \ldots, \frac{N_1 - 21}{6} \) (omit if \( N_1 = 15 \)),

\[
\left( 0, \frac{5N_1 - 63}{12} \right), \left( \frac{(4k + 5)N_1 - 39}{12} + r \right)
\]
for \( r = 0, 1, \ldots, \frac{N_1 - 27}{12} \) (omit if \( N_1 = 15 \)),

\[
\left( 0, \frac{7N_1 - 57}{12} + r \right), \left( \frac{(6k + 7)N_1 - 69}{12} \right)
\]
for \( r = 0, 1, \ldots, \frac{N_1 - 15}{12} \),

\[
\left( 0, \frac{3N_1 - 29}{4} - r \right), \left( \frac{(4k + 9)N_1 - 51}{12} + r \right)
\]
for \( r = 0, 1, \ldots, \frac{N_1 - 27}{12} \) (omit if \( N_1 = 15 \)),

\[
\left( 0, \frac{11N_1 - 57}{12} + r \right), \left( \frac{(6k + 11)N_1 - 81}{12} - r \right)
\]
for \( r = 0, 1, \ldots, \frac{N_1 - 27}{2} \) (omit if \( N_1 = 15 \)),

\[
\left( 0, \frac{N_1 - 15}{6} \right), \left( \frac{(3k + 2)N_1 - 18}{12} \right), \left( 0, \frac{3N_1 - 21}{4} \right), \left( \frac{(4k + 11)N_1 - 69}{12} \right)
\]
\[
\left( 0, (N_1 - 5), \frac{k + 6}{6} \right)
\]

\[
\left( 0, \frac{3N_1 - 25}{4} \right), \left( \frac{(2k + 5)N_1 - 51}{12} \right)
\]

\[
\left( 0, (N_1 - 1), \frac{(k + 6)N_1 - 18}{6} \right)
\]

and \( \left( 0, \frac{5N_1 - 33}{6} \right), \left( \frac{(3k + 5)N_1 - 33}{6} \right) \).
This collection of blocks along with the base blocks for a cyclic STS($N_1$) on $Z_{N_1} \times \{1\}$ under the automorphism $(0_1, 1_1, \ldots, (N_1 - 1)_1)$ form a complete set of base blocks for a bicyclic STS($\nu$) with $\nu = N_1 + N_2$. □

Lemma 3.7. A bicyclic STS($\nu$) on the set $X$ admitting the automorphism $\pi$ exists if $N_1 \equiv 3 \pmod{12}$, $N_1 > 3$, and $k \equiv 6 \pmod{12}$.

Proof. Consider the following collection of base blocks:

\[
\begin{align*}
&\left(0_2, \left(\frac{kN_1 - 18}{6} - 2r\right)_2, \left(\frac{kN_1 - 2}{2} - r\right)_2\right) \\
&\text{for } r = \frac{N_1 - 9}{6}, \frac{N_1 - 3}{6}, \ldots, \frac{kN_1 - 30}{12}, \\
&\left(0_2, \left(\frac{kN_1 - 24}{6} - 2r\right)_2, \left(\frac{kN_1 - 3}{3} - r\right)_2\right) \text{ for } r = 0, 1, \ldots, \frac{kN_1 - 30}{12}, \\
&\left(0_2, \left(\frac{kN_1}{6}\right)_2, \left(\frac{kN_1 + 3}{3}\right)_2\right), \left(0_2, \left(\frac{kN_1}{3}\right)_2, \left(\frac{2kN_1}{3}\right)_2\right), \\
&\left(0_1, \left(\frac{N_1 - 9}{6} + r\right)_2, \left(\frac{(k+1)N_1 - 15}{6} - r\right)_2\right) \text{ for } r = 0, 1, \ldots, \frac{N_1 - 15}{6}, \\
&\left(0_1, \left(\frac{5N_1 - 51}{12} - r\right)_2, \left(\frac{(4k+5)N_1 - 27}{12} + r\right)_2\right) \text{ for } r = 0, 1, \ldots, \frac{N_1 - 15}{12}, \\
&\left(0_1, \left(\frac{7N_1 - 21}{12} + r\right)_2, \left(\frac{(6k+7)N_1 - 45}{12} - r\right)_2\right) \text{ for } r = 0, 1, \ldots, \frac{N_1 - 27}{12}, \\
&\text{(omit if } N_1 = 15), \\
&\left(0_1, \left(\frac{3N_1 - 13}{4} + r\right)_2, \left(\frac{(2k+3)N_1 - 17}{4} - r\right)_2\right) \text{ for } r = 0, 1, \ldots, \frac{N_1 - 15}{12}, \\
&\left(0_1, \left(\frac{11N_1 - 69}{12} - r\right)_2, \left(\frac{(4k+11)kN_1 - 33}{12} + r\right)_2\right) \text{ for } r = 0, 1, \ldots, \frac{N_1 - 27}{12}, \text{(omit if } N_1 = 15), \\
&\left(0_1, (N_1 - 1)_2, \left(\frac{(k+6)N_1 - 18}{6}\right)_2\right), \left(0_1, (N_1 - 4)_2, \left(\frac{(k+6)N_1 - 12}{6}\right)_2\right), \\
&\left(0_1, \left(\frac{11N_1 - 45}{12}\right)_2, \left(\frac{(5k+11)N_1 - 39}{12}\right)_2\right), \left(0_1, \left(\frac{11N_1 - 57}{12}\right)_2, \right), \left(2k+7\right)N_1 - 33\right)_2, \text{ and } \left(0_1 \left(\frac{N_1 - 5}{2}\right)_2, \left(\frac{(k+1)N_1 - 5}{2}\right)_2\right).
\end{align*}
\]
This collection of blocks along with the base blocks for a cyclic \( \text{STS}(N_1) \) on \( Z_{N_1} \times \{ 1 \} \) under the automorphism \( (0, 1, 1, \ldots, (N_1 - 1)_1) \) form a complete set of base blocks for a bicyclic \( \text{STS}(n) \) with \( \nu = N_1 + N_2 \).  

**Lemma 3.8.** A bicyclic \( \text{STS}(n) \) on the set \( X \) admitting the automorphism \( \pi \) exists if \( N_1 \equiv 9 \pmod{12}, N_1 > 9, \) and \( k \equiv 0 \pmod{12} \).

**Proof.** Consider the following collection of base blocks:

\[
\left( 0_2, \left( \frac{kN_1 + 18}{6} + r \right)_2, \left( \frac{kN_1 - 3}{3} - r \right)_2, \right) \quad \text{for } r = 0, 1, \ldots, \frac{kN_1 - 36}{12},
\]

\[
\left( 0_2, \left( \frac{kN_1 + 6}{3} + r \right)_2, \left( \frac{kN_1 - 2}{2} - r \right)_2, \right)
\]

for \( r = \frac{N_1 - 9}{6}, \frac{N_1 - 3}{6}, \ldots, \frac{N_1 - 24}{12}, \)

\[
\left( 0_2, (N_1 + 1)_2, \left( \frac{kN_1 + 3}{3} \right)_2, \left( \frac{kN_1}{3} \right)_2, \right),
\]

\[
\left( 0_1, \left( \frac{N_1 - 3}{6} + r \right)_2, \left( \frac{(k+1)N_1 - 21}{6} - r \right)_2 \right) \quad \text{for } r = 0, 1, \ldots, \frac{N_1 - 21}{6},
\]

\[
\left( 0_1, \left( \frac{5N_1 - 57}{12} - r \right)_2, \left( \frac{(4k+5)N_1 - 33}{12} + r \right)_2 \right) \quad \text{for } r = 0, 1, \ldots, \frac{N_1 - 21}{12},
\]

\[
\left( 0_1, \left( \frac{2N_1 - 18}{3} - r \right)_2, \left( \frac{(k+1)N_1 - 7}{2} + r \right)_2 \right) \quad \text{for } r = 0, 1, \ldots, \frac{N_1 - 21}{12},
\]

\[
\left( 0_1, \left( \frac{3N_1 - 19}{4} + r \right)_2, \left( \frac{(2k+3)N_1 - 27}{4} - r \right)_2 \right) \quad \text{for } r = 0, 1, \ldots, \frac{N_1 - 21}{12},
\]

\[
\left( 0_1, \left( \frac{11N_1 - 87}{12} - r \right)_2, \left( \frac{(4k+11)N_1 - 51}{12} + r \right)_2 \right) \quad \text{for } r = 0, 1, \ldots, \frac{N_1 - 21}{12},
\]

\[
\left( 0_1, \left( \frac{3N_1 - 23}{4} \right)_2, \left( \frac{(2k+5)N_1 - 45}{12} \right)_2, \left( \frac{N_1 - 15}{6} \right)_2, \left( \frac{(3k+2)N_1 - 18}{12} \right)_2 \right),
\]

\[
\left( 0_1, \left( \frac{11N_1 - 63}{12} \right)_2, \left( \frac{(2k+11)N_1 - 75}{12} \right)_2, \left( \frac{N_1 - 15}{6} \right)_2, \left( \frac{(k+6)N_1 - 18}{6} \right)_2 \right),
\]

\[
\left( 0_1, (N_1 - 4)_2, \left( \frac{k+6)N_1 - 12}{6} \right)_2 \right), \text{ and } \left( 0_1, (N_1 - 4)_2, \left( \frac{(k+2)N_1 - 10}{2} \right)_2 \right).
\]
This collection of blocks along with the base blocks for a cyclic $\text{STS}(N_1)$ on $\mathbb{Z}_{N_1} \times \{1\}$ under the automorphism $(0_1, 1_1, \ldots, (N_1 - 1)_1)$ form a complete set of base blocks for a bicyclic $\text{STS}(v)$ with $v = N_1 + N_2$. □

Lemma 3.9. A bicyclic $\text{STS}(v)$ on the set $X$ admitting the automorphism $\pi$ exists if $N_1 \equiv 9 \pmod{12}$, $N_1 > 9$, and $k \equiv 6 \pmod{12}$.

Proof. Consider the following collection of base blocks:

\[
\begin{align*}
&\left(0_2, \left(\frac{kN_1 - 18}6 - 2r\right)_2, \left(\frac{kN_1 - 2}2 - r\right)_2\right) \\
&\text{for } r = \frac{N_1 - 9}{6}, \frac{N_1 - 3}{6}, \ldots, \frac{kN_1 - 30}{12}, \\
&\left(0_2, \left(\frac{kN_1 - 24}6 - 2r\right)_2, \left(\frac{kN_1 - 3}3 - r\right)_2\right) \text{ for } r = 0, 1, \ldots, \frac{kN_1 - 30}{12}, \\
&\left(0_2, \left(\frac{kN_1}6\right)_2, \left(\frac{kN_1 + 3}3\right)_2\right), \left(0_2, \left(\frac{kN_1}3\right)_2, \left(\frac{2kN_1}3\right)_2\right), \\
&\left(0_1, \left(\frac{N_1 - 3}{6} + r\right)_2, \left(\frac{(k+1)N_1 - 9}{6} - r\right)_2\right) \text{ for } r = 0, 1, \ldots, \frac{N_1 - 9}{6}, \\
&\left(0_1, \left(\frac{5N_1 - 45}{12} - r\right)_2, \left(\frac{(4k+5)N_1 - 21}{12} + r\right)_2\right) \text{ for } r = 0, 1, \ldots, \frac{N_1 - 33}{12}
\end{align*}
\] (omitted if $N_1 = 21$),

\[
\begin{align*}
&\left(0_1, \left(\frac{7N_1 - 63}{12} - r\right)_2, \left(\frac{(4k+7)N_1 - 27}{12} + r\right)_2\right) \text{ for } r = 0, 1, \ldots, \frac{N_1 - 21}{12}, \\
&\left(0_1, \left(\frac{3N_1 - 7}{4} + r\right)_2, \left(\frac{(2k+3)N_1 - 15}{4} - r\right)_2\right) \text{ for } r = 0, 1, \ldots, \frac{N_1 - 21}{12}, \\
&\left(0_1, \left(\frac{11N_1 - 27}{12} + r\right)_2, \left(\frac{(6k+11)N_1 - 63}{12} - r\right)_2\right) \text{ for } r = 0, 1, \ldots, \frac{N_1 - 33}{12}
\end{align*}
\] (omitted if $N_1 = 21$),

\[
\begin{align*}
&\left(0_1, \left(\frac{7N_1 - 39}{12}\right)_2, \left(\frac{(4k+7)N_1 - 51}{12}\right)_2\right), \left(0_1, \left(\frac{3N_1 - 11}{4}\right)_2, \left(\frac{(4k+11)N_1 - 51}{12}\right)_2\right), \\
&\left(0_1, \left(\frac{11N_1 - 39}{12}\right)_2, \left(\frac{(5k+11)N_1 - 33}{12}\right)_2\right), \left(0_1, \left(\frac{N_1 - 1}{2}\right)_2, \left(\frac{(k+6)N_1 - 18}{6}\right)_2\right), \\
&\left(0_1, \left(\frac{N_1 - 4}{2}\right)_2, \left(\frac{(k+6)N_1 - 12}{6}\right)_2\right), \text{ and } \left(0_1, \left(\frac{2N_1 - 9}{3}\right)_2, \left(\frac{(3k+4)N_1 - 18}{6}\right)_2\right).
\end{align*}
\]
This collection of blocks along with the base blocks for a cyclic $\text{STS}(N_1)$ on $\mathbb{Z}_{N_1} \times \{1\}$ under the automorphism $(0_1, 1_1, \ldots, (N_1 - 1)_1)$ form a complete set of base blocks for a bicyclic $\text{STS}(v)$ with $v = N_1 + N_2$. □

Lemmas 3.6–3.9 combine to give us the following theorem.

**Theorem 3.10.** A bicyclic $\text{STS}(v)$ where $v = N_1 + N_2$ and $N_2 = kN_1$ admitting an automorphism whose disjoint cyclic decomposition is a cycle of length $N_1$ and a cycle of length $N_2$ exists if $N_1 \equiv 3 \pmod{6}$, $N_1 > 9$, and $k \equiv 0 \pmod{6}$.

Combining Theorems 3.5 and 3.10 with the previous mentioned results for $N_1 = 3$ and $N_1 = 7$ gives us the necessary and sufficient conditions.

**Theorem 3.11.** A bicyclic $\text{STS}(v)$ where $v = N_1 + N_2$ admitting an automorphism whose disjoint cyclic decomposition is a cycle of length $N_1$, where $N_1 > 1$, and a cycle of length $N_2$ exists if and only if $N_1 \equiv 1 \text{ or } 3 \pmod{6}$, $N_1 \neq 9$, $N_1 \mid N_2$, and $v = N_1 + N_2 \equiv 1 \text{ or } 3 \pmod{6}$.

**References**


