SOME INEQUALITIES FOR THE MAXIMUM MODULUS OF RATIONAL FUNCTIONS

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ABSTRACT. For a polynomial $p(z)$ of degree $n$, it follows from the Maximum Modulus Theorem that $\max_{|z|=R \geq 1} |p(z)| \leq R^n \max_{|z|=1} |p(z)|$. It was shown by Ankeny and Rivlin in 1955 that if $p(z) \neq 0$ for $|z| < 1$ then $\max_{|z|=R \geq 1} |p(z)| \leq \frac{R^n+1}{2} \max_{|z|=1} |p(z)|$. These two results were extended to rational functions by Govil and Mohapatra [4]. In this paper, we give refinements of these results of Govil and Mohapatra.

1. Introduction and Statement of Results

Let $\mathcal{P}_n$ denote the set of all complex algebraic polynomials $p$ of degree at most $n$ and let $p'$ be the derivative of $p$. For a function $f$ defined on the unit circle $T = \{z \mid |z|=1\}$ in the complex plane $\mathbb{C}$, set $\|f\| = \sup_{z \in T} |f(z)|$, the Chebyshev norm of $f$ on $T$.

Let $\mathbb{D}_-$ denote the region strictly inside $T$, and $\mathbb{D}_+$ the region strictly outside $T$. For $a_v \in \mathbb{C}$, $v = 1, 2, \ldots, n$, let $w(z) = \prod_{v=1}^{n} (z-a_v)$, $B(z) = \prod_{v=1}^{n} (1-\overline{a}_v z)/(z-a_v)$ being the Blaschke product, and $\mathcal{R}_n = \mathcal{R}_n(a_1, a_2, \ldots, a_n) = \{p(z)/w(z) \mid p \in \mathcal{P}_n\}$. Then $\mathcal{R}_n$ is the set of rational functions with possible poles at $a_1, a_2, \ldots, a_n$ and having a finite limit at $\infty$. Also note that $B(z) \in \mathcal{R}_n$.

DEFINITIONS.

(i): For polynomial $p(z) = \sum_{v=0}^{n} \alpha_v z^v$, the conjugate transpose (reciprocal) $p^*$ of $p$ is defined by

$$ p^*(z) = z^n \overline{p(1/z)} = z^n \overline{p(1/z)} = \overline{\alpha}_0 z^n + \overline{\alpha}_1 z^{n-1} + \cdots + \overline{\alpha}_n. $$

(ii): For rational function $r(z) = p(z)/w(z) \in \mathcal{R}_n$, the conjugate transpose, $r^*$, of $r$ is defined by

$$ r^*(z) = B(z) \overline{r(1/z)} = B(z) \overline{r(1/z)}. $$

(iii): The polynomial $p \in \mathcal{P}_n$ is self-inversive if $p^*(z) = \lambda p(z)$ for some $\lambda \in T$.

(iv): The rational function $r \in \mathcal{R}_n$ is self-inversive if $r^*(z) = \lambda r(z)$ for some $\lambda \in T$.

It is easy to verify that if $r \in \mathcal{R}_n$ and $r = p/w$, then $r^* = p^*/w$ and hence $r^* \in \mathcal{R}_n$. So $p/w$ is self-inversive if and only if $p$ is self-inversive.

If $p \in \mathcal{P}_n$, then it is well known that

$$ \max_{|z|=R \geq 1} |p(z)| \leq R^n \|p\|. \tag{1.1} $$

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This inequality is an immediate consequence of the Maximum Modulus Theorem. Further, if \( p \) has all its zeros in \( T \cup \mathbb{D}_+ \), then

\[
\max_{|z|=R \geq 1} |p(z)| \leq \frac{R^n + 1}{2} \|p\|.
\]

The inequality (1.2) is due to Ankeny and Rivlin [1]. Both inequalities (1.1) and (1.2) are sharp, inequality (1.1) becomes equality for \( p(z) = \lambda z^n \) where \( \lambda \in \mathbb{C} \), and inequality (1.2) becomes equality for \( p(z) = \alpha z^n + \beta \) where \( |\alpha| = |\beta| \).

Govil and Mohapatra [4] gave a result analogous to inequality (1.1), but for rational functions, as follows.

**THEOREM A.** If \( r(z) = \frac{p(z)}{w(z)} = \frac{p(z)}{\prod_{v=1}^{n} (z - a_v)} \in \mathbb{R}_n \) is a rational function with \( |a_v| > 1 \) for \( 1 \leq v \leq n \), then for \( |z| \geq 1 \),

\[
|r(z)| \leq \|r\| \|B(z)\|.
\]

This result is best possible and equality holds for \( r(z) = \lambda \prod_{v=1}^{n} \frac{1 - \bar{a}_v z}{z - a_v} = \lambda B(z) \) where \( \lambda \in \mathbb{C} \).

In the same paper, Govil and Mohapatra [4] also proved a result given below, that is analogous to inequality (1.2) for rational functions.

**THEOREM B.** Let \( r(z) = \frac{p(z)}{w(z)} = \frac{p(z)}{\prod_{v=1}^{n} (z - a_v)} \in \mathbb{R}_n \) with \( |a_v| > 1 \) for \( 1 \leq v \leq n \). If all the zeros of \( r \) lie in \( T \cup \mathbb{D}_+ \), then for \( |z| \geq 1 \),

\[
|r(z)| \leq \|r^*\| \|B(z)\| + \frac{1}{2}.
\]

This result is best possible and equality holds for the rational function \( r(z) = \alpha B(z) + \beta \) where \( |\alpha| = |\beta| \).

In this paper we prove the following refinements of the above two theorems. Here \( p(z) = \sum_{v=0}^{n} \alpha_v z^v \) is a polynomial of degree \( n \).

**Theorem 1.1.** If

\[
r(z) = \frac{p(z)}{w(z)} = \frac{p(z)}{\prod_{v=1}^{n} (z - a_v)} \in \mathbb{R}_n
\]

is a rational function with \( |a_v| > 1 \), \( 1 \leq v \leq n \), then for \( |z| \geq 1 \),

\[
|r(z)| \leq \|r\| \|B(z)\| \left( 1 - \frac{|r^*(0)| (|z| - 1)}{|r^*(0)| + |z| \|r^*\|} \right).
\]

The result is best possible and equality holds for \( r(z) = \lambda B(z) \) where \( \lambda \in \mathbb{C} \).

It is clear that Theorem 1 sharpens Theorem A. Also, we can use Theorem 1 to derive a sharpening form of Bernstein’s Inequality for polynomials. For this, let
\[ p(z) = \sum_{v=0}^{n} \alpha_v z^v \] be a polynomial of degree \( n \). Then \( r(z) = \frac{p(z)}{\prod_{v=1}^{n} (z - a_v)} \in \mathcal{R}_n \)
and hence by Theorem 1, for \( |z| \geq 1 \),
\begin{equation}
\left| \frac{r(z)}{B(z)} \right| = \left| \frac{p(z)}{\prod_{v=1}^{n} (1 - \alpha_v z)} \right| \leq \left\| r \right\| \left\{ 1 \left( \left\| r \right\| - |r^*(0)| (|z| - 1) \right) \right\}.
\end{equation}
If \( z^* \) on \( |z| = 1 \) is such that
\begin{equation}
\left\| r \right\| = |r(z^*)| = \frac{|p(z^*)|}{\prod_{v=1}^{n} (z^* - a_v)}
\end{equation}
then we get from (1.6)
\begin{equation}
\left| \frac{p(z)}{\prod_{v=1}^{n} (1 - \alpha_v z)} \right| \leq \frac{|p(z^*)|}{\prod_{v=1}^{n} |z^* - a_v|} \left\{ 1 \left( \frac{|p(z^*)| - |r^*(0)| \prod_{v=1}^{n} |z^* - a_v|}{|r^*(0)| \prod_{v=1}^{n} |z^* - a_v| + |z| |p(z^*)|} \right) \right\}.
\end{equation}
Since \( p(z) = \sum_{v=0}^{n} \alpha_v z^v \) and \( r^*(z) = \frac{p^*(z)}{\prod_{v=1}^{n} (z - a_v)} \), we get \( |r^*(0)| = \frac{|\alpha_n|}{\prod_{v=1}^{n} |a_v|} \)
and therefore from (1.8) we have for \( |z| > 1 \),
\begin{equation}
|p(z)| \leq |p(z^*)| |z|^n \left\{ 1 \left( \frac{|p(z^*)| - |\alpha_n| \prod_{v=1}^{n} |z^* - a_v|}{|\alpha_n| \prod_{v=1}^{n} |z^* - a_v| + |z| |p(z^*)|} \right) \right\}.
\end{equation}
Since (1.9) holds for all \( |a_v| \geq 1 \), where \( 1 \leq v \leq n \), making \( |a_v| \to \infty \), where
\begin{equation}
|p(z)| \leq \left\| r \right\| |z|^n \left\{ 1 \left( \frac{\left\| r \right\| - |\alpha_n| (|z| - 1)}{|\alpha_n| + |z| \left\| r \right\|} \right) \right\},
\end{equation}
We show in Lemma 5 in the next section that (1.10) implies for \( |z| \geq 1 \)
\begin{equation}
|p(z)| \leq \left\| r \right\| |z|^n \left\{ 1 \left( \frac{\left\| r \right\| - |\alpha_n| (|z| - 1)}{|\alpha_n| + |z| \left\| r \right\|} \right) \right\},
\end{equation}
which is equivalent to that for \( |z| = R \geq 1 \),
\begin{equation}
|p(z)| \leq R^n \left\{ 1 \left( \frac{\left\| r \right\| - |\alpha_n| (R - 1)}{|\alpha_n| + R \left\| r \right\|} \right) \right\} \left\| r \right\|.
\end{equation}
This rate of growth result for a polynomial, which is a sharpening of Bernstein Inequality, first appeared as Lemma 3 of [2].
As a refinement of Theorem B, we shall prove

**Theorem 1.2.** Let
\[ r(z) = \frac{p(z)}{w(z)} = \frac{p(z)}{\prod_{v=1}^{n} (z - a_v)} \in \mathcal{R}_n \]
with \( |a_v| > 1 \) for \( 1 \leq v \leq n \). If all the zeros of \( r \) lie in \( \mathbb{T} \cup \mathbb{D}_+ \), then for \( |z| \geq 1 \)
\[ |r(z)| \leq \frac{1}{2} \left( \left\| r \right\| (|B(z)| + 1) - (\left\| B(z) \right\| - 1 \min_{|z|=1} |r(z)|) \right). \]
Clearly Theorems 1.1 and 1.2 without any additional hypotheses, give bounds that are sharper than those obtainable from Theorems A and B respectively.
2. Lemmas

The following is a well known generalization of Schwarz’s Lemma (see, for example, [3]).

Lemma 2.1. If \( f \) is analytic inside and on the circle \(|z| = 1\), then for \(|z| \leq 1\),

\[
|f(z)| \leq \|f\| \|\frac{|f|}{|f(0)|} + \|f\|.
\]

The next two results are due to Govil and Mohapatra [4].

Lemma 2.2. Let \( r \in \mathcal{R}_n \) with all its poles in \( \mathbb{D}_+ \). If \( r \) has all its zeros in \( \mathbb{T} \cup \mathbb{D}_+ \),
then for all \(|z| \geq 1\), \(|r(z)| \leq |r^*(z)|\).

Lemma 2.3. Let \( r \in \mathcal{R}_n \) with all its poles in \( \mathbb{D}_+ \). Then for \(|z| \geq 1\),
\[
|r(z)| + |r^*(z)| \leq \|r\|(|B(z)| + 1).
\]

Lemma 2.4. Let \( r \in \mathcal{R}_n \) with all its poles in \( \mathbb{D}_+ \). If \( r \) has all its zeros in \( \mathbb{T} \cup \mathbb{D}_+ \),
then for \(|z| \geq 1\), we have
\[
|r(z)| + (|B(z)| - 1) \min_{|z| = 1} |r(z)| \leq |r^*(z)|.
\]

Proof. Since the rational function \( r \) has no zeros in \( \mathbb{D}_- \) hence for every \( \alpha \in \mathbb{C} \) with \(|\alpha| < 1\), the rational function \( r(z) - \alpha \min_{|z| = 1} |r(z)| \) has no zero in \( \mathbb{D}_- \) and has all its poles, like \( r \), in \( \mathbb{D}_+ \). Applying Lemma 2.2 to \( r(z) - \alpha \min_{|z| = 1} |r(z)| \) we get that for \(|z| \geq 1\)
\[
|r(z) - \alpha \min_{|z| = 1} |r(z)|| \leq |r^*(z) - B(z)\alpha \min_{|z| = 1} |r(z)||,
\]
and so for \(|z| \geq 1\),
\[
|r(z) - |\alpha| \min_{|z| = 1} |r(z)|| \leq |r^*(z) - B(z)\alpha \min_{|z| = 1} |r(z)||.
\]

With the appropriate choice of \( \arg(\alpha) \) we then have for \(|z| \geq 1\),
\[
|r(z)| - |\alpha| \min_{|z| = 1} |r(z)| \leq |r^*(z)| - |\alpha| |B(z)| \min_{|z| = 1} |r(z)|.
\]

Note that \( r \) has no zeros in \( \mathbb{D}_- \) and so is analytic in \(|z| \leq 1\). Hence by the Minimum Modulus Theorem, we have \(|r(z)| > |\alpha| \min_{|z| = 1} |r(z)|\) for \(|z| \leq 1\). Therefore for \(|z| \geq 1\) we get
\[
|r^*(z)| = |\frac{B(z)r(1/\overline{z})}{|B(z)|}\overline{r(1/\overline{z})}| > |\alpha| |B(z)| \min_{|z| = 1} |r(z)|,
\]
which clearly implies that the right-hand side of (2.2) is positive. Making \(|\alpha| \to 1\)
in (2.3), we easily get
\[
|r(z)| + (|B(z)| - 1) \min_{|z| = 1} |r(z)| \leq |r^*(z)|, \text{ for } |z| \geq 1,
\]
which is (2.2), and thus the proof of Lemma 2.4 is complete. \(\square\)
Lemma 2.5. The function
\[ g(x) = x \left\{ 1 - \frac{(x - |\alpha_n|)(|z| - 1)}{|\alpha_n| + |z|x} \right\}, \]
where \( \alpha_n, z \in \mathbb{C} \) with \( z \neq 0 \), is an increasing function for \( x \geq 0 \).

Proof. We have
\[ g'(x) = \frac{|z|x^2 + 2|\alpha_n||z| + |z|^2|\alpha_n|^2}{(|\alpha_n| + |z|x)^2} \geq 0 \]
for \( x \geq 0 \). So \( g \) is an increasing function for \( x \geq 0 \), as claimed. \( \square \)

3. Proofs of Theorems

Proof of Theorem 1.1. Since
\[ r(z) = \frac{p(z)}{w(z)} = \frac{p(z)}{\prod_{v=1}^n(z - a_v)} \in \mathcal{R}_n \]
with \( |a_v| > 1 \) for \( 1 \leq v \leq n \), the function \( r^*(z) = p^*(z)/\prod_{v=1}^n(z - a_v) \) is analytic in \( |z| \leq 1 \). Therefore by Lemma 2.1 we get that, for \( |z| \leq 1 \),
\[ |r^*(z)| \leq ||r^*|| \frac{||r^*|| |z| + |r^*(0)|}{|r^*(0)||z| + ||r^*||} \]
and since \( ||r^*|| = ||r|| \), inequality (3.1) is in fact equivalent to the inequality that, for \( |z| \leq 1 \),
\[ |r^*(z)| \leq ||r|| \frac{||r|| |z| + |r^*(0)|}{|r^*(0)||z| + ||r||}. \]
Since by definition \( r^*(z) = B(z)r(1/\overline{z}) \), we get from (3.2) that for \( |z| \leq 1 \),
\[ \overline{r(1/\overline{z})} \leq \frac{||r||}{B(z)} \frac{||r|| |z| + |r^*(0)|}{|r^*(0)||z| + ||r||}, \]
which clearly gives that for \( |z| \geq 1 \),
\[ |r(z)| \leq \frac{||r||}{B(1/\overline{z})} \frac{||r|| + |r^*(0)| |z|}{|r^*(0)||z| + ||r|| |z|}. \]
It is clear from the definition of \( B(z) \) that \( |B(1/\overline{z})| = 1/|B(z)| \) and this, when combined with (3.3), gives that for \( |z| \geq 1 \),
\[ |r(z)| \leq ||r|| \frac{||r|| + |r^*(0)| |z|}{|r^*(0)||z| + ||r|| |z|} \]
\[ = ||r|| |B(z)| \left( 1 - \frac{(||r|| - |r^*(0)|)(|z| - 1)}{|r^*(0)||z| + ||r|| |z|} \right), \]
which is (1.5) and this completes the proof of the Theorem 1.1. \( \square \)

Proof of Theorem 1.2. Since \( r \in \mathcal{R}_n \) and has all its poles in \( \mathbb{D}_+ \) hence, by Lemma 2.3, for \( |z| \geq 1 \) we have
\[ |r(z)| + |r^*(z)| \leq ||r||(|B(z)| + 1). \]
Because $r$ has all its zeros in $\mathbb{T} \cup \mathbb{D}_+$, therefore we can apply Lemma 2.4 to $r$, and this will give that for $|z| \geq 1$,
\begin{equation}
|r(z)| + (|B(z)| - 1) \min_{|z|=1} |r(z)| \leq |r^*(z)|.
\end{equation}
Combining the conclusion of (3.5) with (3.4) we get that for $|z| \geq 1$.
\[2|r(z)| + (|B(z)| - 1) \min_{|z|=1} |r(z)| \leq \|r\|(|B(z)| + 1),\]
which is clearly equivalent to
\[|r(z)| \leq \frac{1}{2} \left( \|r\|(|B(z)| + 1) - (|B(z)| - 1) \min_{|z|=1} |r(z)| \right),\]
and the proof of Theorem 1.2 is thus complete. \square

References