# The Number of Zeros in a Disk for a Certain Class of Polynomials

Robert Gardner and Brett Shields

*Department of Mathematics and Statistics*

*East Tennessee State University*

*Johnson City, Tennessee* $37614 - 0663$

**Abstract.** In this paper, we consider the class $\mathcal{P}_{n,\mu}$, of polynomials of the form $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$. We impose hypotheses on the coefficients of such a polynomial and then restrict the number of zeros in a disk centered at the origin.

*Mathematics subject classification* (2010): 30A99, 30E10.

*Keywords and phrases*: Location of zeros, number of zeros, zeros of polynomials.

## 1 Introduction

While studying Bernstein type inequalities, Chan and Malik [5] introduced the class of polynomials of the form $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$. We denote the class of all of such polynomials as $\mathcal{P}_{n,\mu}$. Notice that when $\mu = 1$, we simply have the class of all polynomials of degree $n$. This class has been extensively studied in connection with Bernstein type inequalities (see, for example, [3, 8, 9, 15, 10]).

The purpose of this paper is to study the number of zeros in a disk of a polynomial in the class $\mathcal{P}_{n,\mu}$. The first result concerning counting zeros which is of relevance to our study can be found in Titchmarsh's *The Theory of Functions* [16]:

**Theorem 1.1** *If all of the zeros of polynomial* $p(z) = \sum_{k=0}^{n} a_k z^k$, $a_0 \neq 0$, *lie in* $|z| \leq R$ *and* $|p(z)| \leq M$ *for* $|z| \leq R$, *then the number of zeros in* $|z| \leq \delta R$, *where* $0 < \delta < 1$, *does not exceed*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}.$$

## 2 Results

In this section, we provide three main theorems and several corollaries. We consider polynomials in $\mathcal{P}_{n,\mu}$ and put restrictions on the moduli of the coefficients (in Theorem 2.1), the real parts of the coefficients (in Theorem 2.5), and on both the real and imaginary parts of the coefficients (in Theorem 2.9). Each result puts a bound on the number of zeros of the polynomial in a disk centered about 0.

**Theorem 2.1** *Let* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ *where* $a_0 \neq 0$ *and for some* $t > 0$ *and some* $k$ *with* $\mu \leq k \leq n$,

$$t^\mu |a_\mu| \leq \cdots \leq t^{k-1}|a_{k-1}| \leq t^k |a_k| \geq t^{k+1}|a_{k+1}| \geq \cdots \geq t^{n-1}|a_{n-1}| \geq t^n |a_n|$$

*and* $|\arg a_j - \beta| \leq \alpha \leq \pi/2$ *for* $\mu \leq j \leq n$ *and for some real* $\alpha$ *and* $\beta$. *Then for* $0 < \delta < 1$ *the number of zeros of* $P(z)$ *in the disk* $|z| \leq \delta t$ *does not exceed*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}$$

*where* $M = 2|a_0|t + |a_\mu| t^{\mu+1}(1 - \cos\alpha - \sin\alpha) + 2|a_k| t^{k+1} \cos\alpha + |a_n| t^{n+1}(1 - \cos\alpha - \sin\alpha) + 2\sum_{j=\mu}^{n} |a_j| t^{j+1} \sin\alpha$.

With $t = 1$ in Theorem 2.1 we get the following.

**Corollary 2.2** *Let* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ *where* $a_0 \neq 0$ *and for some* $t > 0$ *and some* $k$ *with* $\mu \leq k \leq n$,

$$|a_\mu| \leq \cdots \leq |a_{k-1}| \leq |a_k| \geq |a_{k+1}| \geq \cdots \geq |a_{n-1}| \geq |a_n|$$

*and* $|\arg a_j - \beta| \leq \alpha \leq \pi/2$ *for* $\mu \leq j \leq n$ *and for some real* $\alpha$ *and* $\beta$. *Then for* $0 < \delta < 1$ *the number of zeros of* $P(z)$ *in the disk* $|z| \leq \delta$ *does not exceed*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}$$

*where* $M = 2|a_0| + |a_\mu|(1 - \cos\alpha - \sin\alpha) + 2|a_k| \cos\alpha + |a_n|(1 - \cos\alpha - \sin\alpha) + 2\sum_{j=\mu}^{n} |a_j| \sin\alpha$.

With $k = n$ in Corollary 2.2 we get:

**Corollary 2.3** *Let* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ *where* $a_0 \neq 0$,

$$|a_\mu| \leq \cdots \leq |a_{n-1}| \leq |a_n|$$

*and* $|arg\, a_j - \beta| \leq \alpha \leq \pi/2$ *for* $\mu \leq j \leq n$ *and for some real* $\alpha$ *and* $\beta$. *Then for* $0 < \delta < 1$ *the number of zeros of* $P(z)$ *in the disk* $|z| \leq \delta$ *does not exceed*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}$$

*where* $M = 2|a_0| + |a_\mu|(1 - \cos\alpha - \sin\alpha) + |a_n|(1 + \cos\alpha - \sin\alpha) + 2\sum_{j=\mu}^{n} |a_j| \sin\alpha$.

With $k = \mu$ in Corollary 2.2 we get:

**Corollary 2.4** *Let* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ *where* $a_0 \neq 0$,

$$|a_\mu| \geq \cdots \geq |a_{n-1}| \geq |a_n|$$

*and* $|arg\, a_j - \beta| \leq \alpha \leq \pi/2$ *for* $\mu \leq j \leq n$ *and for some real* $\alpha$ *and* $\beta$. *Then for* $0 < \delta < 1$ *the number of zeros of* $P(z)$ *in the disk* $|z| \leq \delta$ *does not exceed*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}$$

*where* $M = 2|a_0| + |a_\mu|(1 + \cos\alpha - \sin\alpha) + |a_n|(1 - \cos\alpha - \sin\alpha) + 2\sum_{j=\mu}^{n} |a_j| \sin\alpha$.

Theorem 2.1 requires the moduli of the coefficients of a polynomial in $\mathcal{P}_{n,\mu}$ to satisfy the monotonicity condition as stated in the theorem. We now modify the hypotheses of Theorem 2.1 by imposing the monotonicity condition only on the real parts of the coefficients.

**Theorem 2.5** *Let* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ *where* $a_0 \neq 0$, $Re\, a_j = \alpha_j$ *and* $Im\, a_j = \beta_j$ *for* $\mu \leq j \leq n$. *Suppose that for some* $t > 0$ *and some* $k$ *with* $\mu \leq k \leq n$ *we have*

$$t^\mu \alpha_\mu \leq \cdots \leq t^{k-1}\alpha_{k-1} \leq t^k \alpha_k \geq t^{k+1}\alpha_{k+1} \geq \cdots \geq t^{n-1}\alpha_{n-1} \geq t^n \alpha_n.$$

*Then for* $0 < \delta < 1$ *the number of zeros of* $P(z)$ *in the disk* $|z| \leq \delta t$ *does not exceed*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}.$$

*where* $M = 2(|\alpha_0| + |\beta_0|)t + (|\alpha_\mu| - \alpha_\mu)t^{\mu+1} + 2\alpha_k t^{k+1} + (|\alpha_n| - \alpha_n)t^{n+1} + 2\sum_{j=\mu}^{n} |\beta_j| t^{j+1}$.

With $t = 1$ in Theorem 2.5, we get:

**Corollary 2.6** *Let* $P(z) = a_0 + \displaystyle\sum_{j=\mu}^{n} a_j z^j$ *where* $a_0 \neq 0$, $Re\, a_j = \alpha_j$ *and* $Im\, a_j = \beta_j$ *for* $\mu \leq j \leq n$. *Suppose that we have*

$$\alpha_\mu \leq \cdots \leq \alpha_{k-1} \leq \alpha_k \geq \alpha_{k+1} \geq \cdots \geq \alpha_{n-1} \geq \alpha_n.$$

*Then for* $0 < \delta < 1$ *the number of zeros of* $P(z)$ *in the disk* $|z| \leq \delta$ *does not exceed*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}.$$

*where* $M = 2(|\alpha_0| + |\beta_0|) + (|\alpha_\mu| - \alpha_\mu) + 2\alpha_k + (|\alpha_n| - \alpha_n) + 2\sum_{j=\mu}^{n} |\beta_j|$.

**Example.** Consider the polynomial $P(z) = 0.1 + 0.001z^2 + 2z^3 + 0.002z^4 + 0.002z^5 + 0.001z^6$. The zeros of $P$ are approximately $z_1 = -0.368602$, $z_2 = 0.184076 + 0.319010i$, $z_3 = 0.184076 - 0.319010i$, and $z_4 = 5.62344 + 10.92507i$, $z_5 = 5.62344 - 10.92507i$, and $z_6 = -13.2464$. Corollary 2.6 applies to $P$ with $\mu = 2$ and $k = 3$. With $\delta = 0.37$ we see that it predicts no more than 3.75928 zeros in $|z| \leq 0.37$. In other words, Corollary 2.6 predicts at most three zeros in $|z| \leq 0.37$ In fact, $P$ does have exactly three zeros in $|z| \leq 0.37$, namely $z_1$, $z_2$, and $z_3$. So Corollary 2.6 is sharp for this example.

With $k = n$ in Corollary 2.6 we get:

**Corollary 2.7** *Let* $P(z) = a_0 + \displaystyle\sum_{j=\mu}^{n} a_j z^j$ *where* $a_0 \neq 0$, $Re\, a_j = \alpha_j$ *and* $Im\, a_j = \beta_j$ *for* $\mu \leq j \leq n$. *Suppose that we have*

$$\alpha_\mu \leq \cdots \leq \alpha_{n-1} \leq \alpha_n.$$

*Then for* $0 < \delta < 1$ *the number of zeros of* $P(z)$ *in the disk* $|z| \leq \delta$ *does not exceed*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}.$$

*where* $M = 2(|\alpha_0| + |\beta_0|) + (|\alpha_\mu| - \alpha_\mu) + (|\alpha_n| + \alpha_n) + 2\sum_{j=\mu}^{n} |\beta_j|$.

With $k = \mu$ in Corollary 2.6 we get:

**Corollary 2.8** *Let* $P(z) = a_0 + \displaystyle\sum_{j=\mu}^{n} a_j z^j$ *where* $a_0 \neq 0$, $Re\, a_j = \alpha_j$ *and* $Im\, a_j = \beta_j$ *for* $\mu \leq j \leq n$. *Suppose that we have*

$$\alpha_\mu \geq \cdots \geq \alpha_{n-1} \geq \alpha_n.$$

*Then for* $0 < \delta < 1$ *the number of zeros of* $P(z)$ *in the disk* $|z| \leq \delta$ *does not exceed*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}.$$

*where* $M = 2(|\alpha_0| + |\beta_0|) + (|\alpha_\mu| + \alpha_\mu) + (|\alpha_n| - \alpha_n) + 2\sum_{j=\mu}^{n} |\beta_j|$.

We now state a final theorem in the same style as Theorems 2.1 and 2.5. If we have a monotonicity condition on both the real and imaginary parts of the coefficients (separately), then we have the potential of improving on Theorem 2.5. This leads us to the following.

**Theorem 2.9** *Let* $P(z) = a_0 + \displaystyle\sum_{j=\mu}^{n} a_j z^j$ *where* $a_0 \neq 0$, $Re\, a_j = \alpha_j$ *and* $Im\, a_j = \beta_j$ *for* $\mu \leq j \leq n$. *Suppose that for some* $t > 0$ *and some* $k$ *with* $\mu \leq k \leq n$ *we have*

$$t^\mu \alpha_\mu \leq \cdots \leq t^{k-1}\alpha_{k-1} \leq t^k \alpha_k \geq t^{k+1}\alpha_{k+1} \geq \cdots \geq t^{n-1}\alpha_{n-1} \geq t^n \alpha_n$$

*and for some* $\mu \leq \ell \leq n$ *we have*

$$t^\mu \beta_\mu \leq \cdots \leq t^{\ell-1}\beta_{\ell-1} \leq t^\ell \beta_\ell \geq t^{\ell+1}\beta_{\ell+1} \geq \cdots \geq t^{n-1}\beta_{n-1} \geq t^n \beta_n.$$

*Then for* $0 < \delta < 1$ *the number of zeros of* $P(z)$ *in the disk* $|z| \leq \delta t$ *does not exceed*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}.$$

*where* $M = 2(|\alpha_0| + |\beta_0|)t + (|\alpha_\mu| - \alpha_\mu + |\beta_\mu| - \beta_\mu)t^{\mu+1} + 2(\alpha_k t^{k+1} + \beta_\ell t^{\ell+1}) + (|\alpha_n| - \alpha_n + |\beta_n| - \beta_n)t^{n+1}$.

In Theorem 2.9 if we let $t = 1$ we get the following.

**Corollary 2.10** *Let* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ *where* $a_0 \neq 0$, $Re\, a_j = \alpha_j$ *and* $Im\, a_j = \beta_j$ *for* $\mu \leq j \leq n$. *Suppose that for some* $k$ *with* $\mu \leq k \leq n$ *we have*

$$\alpha_\mu \leq \cdots \leq \alpha_{k-1} \leq \alpha_k \geq \alpha_{k+1} \geq \cdots \geq \alpha_{n-1} \geq \alpha_n$$

*and for some* $\mu \leq \ell \leq n$ *we have*

$$\beta_\mu \leq \cdots \leq \beta_{\ell-1} \leq \beta_\ell \geq \beta_{\ell+1} \geq \cdots \geq \beta_{n-1} \geq \beta_n.$$

*Then for* $0 < \delta < 1$ *the number of zeros of* $P(z)$ *in the disk* $|z| \leq \delta$ *does not exceed*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}.$$

*where* $M = 2(|\alpha_0| + |\beta_0|) + (|\alpha_\mu| - \alpha_\mu + |\beta_\mu| - \beta_\mu) + 2(\alpha_k + \beta_\ell) + (|\alpha_n| - \alpha_n + |\beta_n| - \beta_n)$.

In Corollary 2.10 if we let $k = \ell = n$ we get the following.

**Corollary 2.11** *Let* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ *where* $a_0 \neq 0$, $Re\, a_j = \alpha_j$ *and* $Im\, a_j = \beta_j$ *for* $\mu \leq j \leq n$. *Suppose that for some* $k$ *with* $\mu \leq k \leq n$ *we have*

$$\alpha_\mu \leq \cdots \leq \alpha_{n-1} \leq \alpha_n$$

*and for some* $\mu \leq \ell \leq n$ *we have*

$$\beta_\mu \leq \cdots \leq \beta_{n-1} \leq \beta_n.$$

*Then for* $0 < \delta < 1$ *the number of zeros of* $P(z)$ *in the disk* $|z| \leq \delta$ *does not exceed*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}.$$

*where* $M = 2(|\alpha_0| + |\beta_0|) + (|\alpha_\mu| - \alpha_\mu + |\beta_\mu| - \beta_\mu) + (|\alpha_n| + \alpha_n + |\beta_n| + \beta_n)$.

In Corollary 2.10 if we let $k = \ell = \mu$ we get the following.

**Corollary 2.12** *Let* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ *where* $a_0 \neq 0$, $Re\, a_j = \alpha_j$ *and* $Im\, a_j = \beta_j$ *for* $\mu \leq j \leq n$. *Suppose that for some* $k$ *with* $\mu \leq k \leq n$ *we have*

$$\alpha_\mu \geq \cdots \geq \alpha_{n-1} \geq \alpha_n$$

*and for some* $\mu \leq \ell \leq n$ *we have*

$$\beta_\mu \geq \cdots \geq \beta_{n-1} \geq \beta_n.$$

*Then for* $0 < \delta < 1$ *the number of zeros of* $P(z)$ *in the disk* $|z| \leq \delta$ *does not exceed*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}.$$

*where* $M = 2(|\alpha_0| + |\beta_0|) + (|\alpha_\mu| + \alpha_\mu + |\beta_\mu| + \beta_\mu) + (|\alpha_n| - \alpha_n + |\beta_n| - \beta_n)$.

In Corollary 2.10, we get similar corollaries by letting $k = n$ and $\ell = \mu$, or $k = \mu$ and $\ell = n$.

# 3 Proof of the Results

The following is due to Govil and Rahman and appears in [11].

**Lemma 3.1** *Let* $z, z' \in \mathbb{C}$ *with* $|z| \geq |z'|$. *Suppose* $|arg\, z^* - \beta| \leq \alpha \leq \pi/2$ *for* $z^* \in \{z, z'\}$ *and for some real* $\alpha$ *and* $\beta$. *Then*

$$|z - z'| \leq (|z| - |z'|) \cos \alpha + (|z| + |z'|) \sin \alpha.$$

We now give proofs of our results.

**Proof of Theorem 2.1.** Consider

$$\begin{aligned} F(z) &= (t-z)P(z) = (t-z)(a_0 + \sum_{j=\mu}^{n} a_j z^j) = a_0 t + \sum_{j=\mu}^{n} a_j t z^j - a_0 z - \sum_{j=\mu}^{n} a_j z^{j+1} \\ &= a_0(t-z) + \sum_{j=\mu}^{n} a_j t z^j - \sum_{j=\mu+1}^{n+1} a_{j-1} z^j \\ &= a_0(t-z) + a_\mu t z^\mu + \sum_{j=\mu+1}^{n} (a_j t - a_{j-1}) z^j - a_n z^{n+1}. \end{aligned}$$

For $|z| = t$ we have

$$
\begin{aligned}
|F(z)| &\leq 2|a_0|t + |a_\mu|t^{\mu+1} + \sum_{j=\mu+1}^{n} |a_j t - a_{j-1}|t^j + |a_n|t^{n+1} \\
&= 2|a_0|t + |a_\mu|t^{\mu+1} + \sum_{j=\mu+1}^{k} |a_j t - a_{j-1}|t^j + \sum_{j=k+1}^{n} |a_{j-1} - a_j t|t^j + |a_n|t^{n+1} \\
&\leq 2|a_0|t + |a_\mu|t^{\mu+1} + \sum_{j=\mu+1}^{k} \{(|a_j|t - |a_{j-1}|)\cos\alpha + (|a_{j-1}| + |a_j|t)\sin\alpha\}\, t^j \\
&\quad + \sum_{j=k+1}^{n} \{(|a_{j-1}| - |a_j|t)\cos\alpha + (|a_j|t + |a_{j-1}|)\sin\alpha\}\, t^j + |a_n|t^{n+1} \\
&\quad \text{by Lemma 3.1 with } z = a_j t \text{ and } z' = a_{j-1} \text{ when } 1 \leq j \leq k, \\
&\quad \text{and with } z = a_{j-1} \text{ and } z' = a_j t \text{ when } k+1 \leq j \leq n \\
&= 2|a_0|t + |a_\mu|t^{\mu+1} + \sum_{j=\mu+1}^{k} |a_j|t^{j+1}\cos\alpha - \sum_{j=\mu+1}^{k} |a_{j-1}|t^j\cos\alpha + \sum_{j=\mu+1}^{k} |a_{j-1}|t^j\sin\alpha \\
&\quad + \sum_{j=\mu+1}^{k} |a_j|t^{j+1}\sin\alpha + \sum_{j=k+1}^{n} |a_{j-1}|t^j\cos\alpha - \sum_{j=k+1}^{n} |a_j|t^{j+1}\cos\alpha \\
&\quad + \sum_{j=k+1}^{n} |a_j|t^{j+1}\sin\alpha + \sum_{j=k+1}^{n} |a_{j-1}|t^j\sin\alpha + |a_n|t^{n+1} \\
&= 2|a_0|t + |a_\mu|t^{\mu+1} - |a_\mu|t^{\mu+1}\cos\alpha + |a_k|t^{k+1}\cos\alpha + |a_\mu|t^{\mu+1}\sin\alpha \\
&\quad + |a_k|t^{k+1}\sin\alpha + 2\sum_{j=\mu+1}^{k-1} |a_j|t^{j+1}\sin\alpha + |a_k|t^{k+1}\cos\alpha - |a_n|t^{n+1}\cos\alpha + |a_k|t^{k+1}\sin\alpha \\
&\quad + |a_n|t^{n+1}\sin\alpha + 2\sum_{j=k+1}^{n-1} |a_j|t^{j+1}\sin\alpha + |a_n|t^{n+1} \\
&= 2|a_0|t + |a_\mu|t^{\mu+1} + |a_\mu|t^{\mu+1}(\sin\alpha - \cos\alpha) + 2\sum_{j=\mu+1}^{n-1} |a_j|t^{j+1}\sin\alpha \\
&\quad + 2|a_k|t^{k+1}\cos\alpha + (\sin\alpha - \cos\alpha + 1)|a_n|t^{n+1} \\
&= 2|a_0|t + |a_\mu|t^{\mu+1}(1 - \cos\alpha - \sin\alpha) + 2|a_k|t^{k+1}\cos\alpha \\
&\quad + |a_n|t^{n+1}(1 - \cos\alpha - \sin\alpha) + 2\sum_{j=\mu}^{n} |a_j|t^{j+1}\sin\alpha \\
&= M.
\end{aligned}
$$

Now $F(z)$ is analytic in $|z| \leq t$, and $|F(z)| \leq M$ for $|z| = t$. So by Theorem A and the Maximum Modulus Theorem, the number of zeros of $F$ (and hence of $P$) in $|z| \leq \delta t$ is less than or equal to

$$\frac{1}{\log 1/\delta} \log \frac{M}{|a_0|}.$$

The theorem follows. ∎

**Proof of Theorem 2.5.** As in the proof of Theorem 2.1,

$$F(z) = (t-z)P(z) = a_0(t-z) + a_\mu t z^\mu + \sum_{j=\mu+1}^{n} (a_j t - a_{j-1}) z^j - a_n z^{n+1},$$

and so

$$\begin{aligned} F(z) &= (\alpha_0 + i\beta_0)(t-z) + (\alpha_\mu + i\beta_\mu) t z^\mu + \sum_{j=\mu+1}^{n} ((\alpha_j + i\beta_j)t - (\alpha_{j-1} + i\beta_{j-1})) z^j \\ &\quad -(\alpha_n + i\beta_n) z^{n+1} \\ &= (\alpha_0 + i\beta_0)(t-z) + (\alpha_\mu + i\beta_\mu) t z^\mu + \sum_{j=\mu+1}^{n} (\alpha_j t - \alpha_{j-1}) z^j + i \sum_{j=\mu+1}^{n} (\beta_j t - \beta_{j-1}) z^j \\ &\quad -(\alpha_n + i\beta_n) z^{n+1}. \end{aligned}$$

For $|z| = t$ we have

$$\begin{aligned} |F(z)| &\leq 2(|\alpha_0| + |\beta_0|)t + (|\alpha_\mu| + |\beta_\mu|)t^{\mu+1} + \sum_{j=\mu+1}^{n} |\alpha_j t - \alpha_{j-1}| t^j + \sum_{j=\mu+1}^{n} (|\beta_j| t + |\beta_{j-1}|) t^j \\ &\quad +(|\alpha_n| + |\beta_n|) t^{n+1} \\ &= 2(|\alpha_0| + |\beta_0|)t + (|\alpha_\mu| + |\beta_\mu|)t^{\mu+1} + \sum_{j=\mu+1}^{k} (\alpha_j t - \alpha_{j-1}) t^j + \sum_{j=k+1}^{n} (\alpha_{j-1} - \alpha_j t) t^j \\ &\quad +|\beta_\mu| t^{\mu+1} + 2 \sum_{j=\mu+1}^{n-1} |\beta_j| t^{j+1} + |\beta_n| t^{n+1} + (|\alpha_n| + |\beta_n|) t^{n+1} \\ &= 2(|\alpha_0| + |\beta_0|)t + (|\alpha_\mu| + |\beta_\mu|)t^{\mu+1} - \alpha_\mu t^{\mu+1} + 2\alpha_k t^{k+1} - \alpha_n t^{n+1} + |\beta_\mu| t^{\mu+1} \\ &\quad +2 \sum_{j=\mu+1}^{n} |\beta_j| t^{j+1} + |\alpha_n| t^{n+1} \\ &= 2(|\alpha_0| + |\beta_0|)t + (|\alpha_\mu| - \alpha_\mu) t^{\mu+1} + 2\alpha_k t^{k+1} + (|\alpha_n| - \alpha_n) t^{n+1} + 2 \sum_{j=\mu}^{n} |\beta_j| t^{j+1} \\ &= M. \end{aligned}$$

The result now follows as in the proof of Theorem 2.1. ∎

**Proof of Theorem 2.9.** As in the proof of Theorem 2.1,

$$F(z)=(t-z)P(z)=a_0(t-z)+a_\mu^t z^\mu+\sum_{j=\mu}^{n}(a_jt-a_{j-1})z^j-a_nz^{n+1},$$

and so

$$\begin{aligned}
F(z) &= (\alpha_0+i\beta_0)(t-z)+(\alpha_\mu+i\beta_\mu)tz^\mu+\sum_{j=\mu+1}^{n}((\alpha_j+i\beta_j)t-(\alpha_{j-1}+i\beta_{j-1}))z^j \\
&\quad -(\alpha_n+i\beta_n)z^{n+1} \\
&= (\alpha_0+i\beta_0)(t-z)+(\alpha_\mu+i\beta_\mu)tz^\mu+\sum_{j=\mu+1}^{n}(\alpha_jt-\alpha_{j-1})z^j+i\sum_{j=\mu+1}^{n}(\beta_jt-\beta_{j-1})z^j \\
&\quad -(\alpha_n+i\beta_n)z^{n+1}
\end{aligned}$$

For $|z|=t$ we have

$$\begin{aligned}
|F(z)| &\leq (|\alpha_0|+|\beta_0|)2t+(|\alpha_\mu|+|\beta_\mu|)t^{\mu+1}+\sum_{j=\mu+1}^{n}|\alpha_jt-\alpha_{j-1}|t^j+\sum_{j=\mu+1}^{n}(|\beta_jt+\beta_{j-1}|)t^j \\
&\quad +(|\alpha_n|+|\beta_n|)t^{n+1} \\
&= 2(|\alpha_0|+|\beta_0|)t+(|\alpha_\mu|+|\beta_\mu|)t^{\mu+1}+\sum_{j=\mu+1}^{k}(\alpha_jt-\alpha_{j-1})t^j+\sum_{j=k+1}^{n}(\alpha_{j-1}-\alpha_jt)t^j \\
&\quad +\sum_{j=\mu+1}^{\ell}(\beta_jt-\beta_{j-1})t^j+\sum_{j=\ell+1}^{n}(\beta_{j-1}-\beta_jt)t^j+(|\alpha_n|+|\beta_n|)t^{n+1} \\
&= 2(|\alpha_0|+|\beta_0|)t+(|\alpha_\mu|+|\beta_\mu|)t^{\mu+1}-\alpha_\mu t^{\mu+1}+2\alpha_kt^{k+1}-\alpha_nt^{n+1}-\beta_\mu t^{\mu+1} \\
&\quad +2\beta_\ell t^{\ell+1}-\beta_nt^{n+1}+(|\alpha_n|+|\beta_n|)t^{n+1} \\
&= 2(|\alpha_0|+|\beta_0|)t+(|\alpha_\mu|-\alpha_\mu+|\beta_\mu|-\beta_\mu)t^{\mu+1}+2(\alpha_kt^{k+1}+\beta_\ell t^{\ell+1}) \\
&\quad +(|\alpha_n|-\alpha_n+|\beta_n|-\beta_n)t^{n+1} \\
&= M.
\end{aligned}$$

The result now follows as in the proof of Theorem 2.1. ∎

# References


[1] A. Abueida and R. Gardner, "Some Generalizations of the Enestron-Kakeya Theorem," in *Theory of Polynomials and Applications: Recent Advances*, ed. K. K. Dewan, New Delhi: Deep & Deep Publications, 2007.

[2] A. Aziz and Q. G. Mohammad, On the Zeros of a Certain Class of Polynomials and Related Entire Functions, *Journal of Mathematical Analysis and Applications*, **75** (1980), 495–502.

[3] A. Aziz and W. M. Shah, An Integral Mean Estimate for a Polynomial, *Indian Journal of Pure Applied Mathematics*, **28**, 1413–1419 (1997).

[4] Aziz, A. and B. A. Zargar, Some Extensions of Eneström-Kakeya Theorem, *Glasnik Matematički*, **31**(51), 239–244 (1996).

[5] Chan, T. and M. Malik, On the Erös-Lax Theorem, *Proceedings of the Indian Academy of Sciences*, **92**, 191–193 (1983).

[6] Cao, J. and R. Gardner, Restrictions on the Zeros of a Polynomial as a Consequence of Conditions on the Coefficients of even Powers and Odd Powers of the Variable, *Journal of Computational and Applied Mathematics*, **155**, 153–163 (2003).

[7] K. K. Dewan, "Extremal Properties and Coefficient Estimates for Polynomials with Restricted Zeros and on Location of Zeros of Polynomials," Ph.D. Thesis, Indian Institutes of Technology, Delhi, 1980.

[8] K. K. Dewan, N. Singh, and A. Mir, Extensions of Some Polynomial Inequalities to the Polar Derivative, *Journal of Mathematical Analysis and Applications*, **352**, 807–815 (2009).

[9] K. K. Dewan, N. Singh, and R. Lal, Inequalities for the Polar Derivatives of a Polynomial, *International Journal of Pure and Applied Matheamtics*, **33**, 109–117 (2006).

[10] N. A. Rather and M. I. Mir, Some Refinements of Inequalities for the Polar Derivative of Polynomials with Restricted Zeros, *International Journal of Pure and Applied Mathematics*, **41**, 1065–1074 (2007).

[11] N. K. Govil and Q. I. Rahman, On the Eneström-Kakeya Theorem, *Tôhoku Mathematics Journal*, **20** (1968), 126–136.

[12] G. V. Milovanović, D. S. Mitrinović, and Th. M. Rassias, *Topics in Polynomials: Extremal Problems, Inequalities, Zeros*, Singapore: World Scientific Publishing, 1994.

[13] Q. G. Mohammad, On the Zeros of the Polynomials, *American Mathematical Monthly*, **72**(6) (1965), 631–633.

[14] M. S. Pukhta, On the Zeros of a Polynomial, *Applied Mathematics*, **2** (2011), 1356–1358.

[15] Qazi, M., On the Maximum Modulus of Polynomials, *Proceedings of the American Mathematical Society*, **115**, 337–343 (1992).

[16] E. C. Titchmarsh, *The Theory of Functions*, 2nd Edition, Oxford University Press, London, 1939.