
Piers Rawling
*Department of Philosophy*
*University of Missouri — St. Louis*
*St. Louis, Missouri 63121-4499*
e-mail: sjprawl@umslvma.umsl.edu

**Commentary by Robert Gardner**

Dr. Rawling states two versions of the two envelopes problem. In both versions, an amount of money is placed in an envelope $O$, a coin is tossed and twice the amount of money in $O$ is placed in envelope $T$ if the coin comes up heads or half the amount of money in $O$ is placed in envelope $O$ is placed in envelope $T$ if the coin comes up tails. As he observes, if the amount of money in $O$ is $10$ (or any known quantity $x$), then the expected amount in $T$ is $1.25 \times 10$ (or $1.25x$ in general). Therefore, in such a situation no "exchange paradox" arises. However, if the amount of money in $O$ is not known (say it is some value $2^n$ where $n > 0$ is an integer), then he argues that this leads to the exchange paradox in which if one holds envelope $O$, then s/he deduces that the expected amount in envelope $T$ is greater, and conversely if one holds envelope $T$, then s/he deduces that the expected amount in envelope $O$ is greater. Since this is clearly contradictory, Dr. Rawling is lead to the conclusion that he must either reject
1. \( P(\text{tails} \mid T \text{ contains } 2^n) = P(\text{heads} \mid T \text{ contains } 2^n) = 0.5, \) or

2. countable additivity.

I propose that a third option exists to resolve the paradox which is perhaps even more elementary.

If we interpret the two envelopes problem to consist of the following events (in order):

1. put an amount of money \( x \) in envelope \( O \),

2. flip a coin,

3. put twice the amount \( x \) in envelope \( T \) if the coin comes up heads, and put half the amount \( x \) in envelope \( T \) if the coin comes up tails,

then the exchange paradox is easily explained. The real problem lies in determining how the amount \( x \) which is to be placed in envelope \( O \) is to be determined. If we are given a probability distribution that describes how the quantity is chosen, then the paradox immediately disappears. Suppose this probability distribution has mean \( \mu \). Then the expected amount in envelope \( O \) is \( \mu \) and the expected amount in \( T \) is \( \frac{5}{4} \mu \). No paradox exists and one should choose envelope \( T \). Notice this can be accomplished over a countable sample space for which the probability of no event is zero and yet we still have countable additivity and finite expected values, as illustrated in the following example.

**Example 1.** Suppose a positive integer \( n \) is chosen according to the probability distribution \( p(n) = \frac{1}{2^n} \) and \( \frac{3}{2} \) is placed in envelope \( O \). A coin is tossed and an amount placed in envelope \( T \) as described above. Then the expected amount in \( O \) is \( \sum_{n=1}^{\infty} \left( \frac{3}{2} \right)^n \left( \frac{1}{2^n} \right) = 3 \) and the expected amount in \( T \) is \( 1.25 \times 3 = \frac{15}{4} \). It is fallacious to argue that the expected value of \( O \) is \( \frac{5}{4} \) times the expected value of \( O \), since the expected value of \( O \) is already determined by the probability distribution.

Of course, one can argue that the game given in Example 1 cannot actually be played since it requires the availability of an unbounded amount of money. In the following example, this problem does not arise.
Example 2. Suppose a positive number between 0 and 1 is chosen according to a uniform distribution (i.e. the probability function if $f(x) = 1$) is placed in envelope $O$. Then the probability that a value from set $A \subset (0, 1)$ is chosen is $\int_A 1$ (for generality, we take the integration to be Lebesgue integration). The expected amount in $O$ is $\int_{(0,1)} x = \frac{1}{2}$. Following the coin toss as described above, the expected amount in envelope $T$ is $\frac{5}{8}$. Again, there is no paradox and one should choose envelope $T$. Notice that in this example, we have countable additivity (since Lebesgue integration is countably additive), although the sample space is uncountable.

Notice that one can argue that $eav(T) = \frac{5}{4} eav(O)$. However, as above, it is not valid to calculate $eav(O)$ in terms of $eav(T)$ as $eav(O) = \frac{e}{4} eav(T)$, since $eav(O)$ is given by the probability distribution. In addition, we still have $p(\text{tails} \mid T \text{ contains } x) = p(\text{heads} \mid T \text{ contains } x) = 0.5$.

Robert B. Gardner  
Department of Mathematics  
Department of Physics and Astronomy  
East Tennessee State University  
Box 70663  
Johnson City, TN 37614  
e-mail: gardnerr@etsu.edu

Rawling’s Reply to Gardner

I thank Professor Gardner for his commentary. I shall here respond to him on two issues, largely for purposes of clarification.

Professor Gardner suggests that

(A) \[ p(\text{tails} \mid T \text{ contains } \$2n) = p(\text{heads} \mid T \text{ contains } \$2n) = 0.5 \]

is consistent with countable additivity. I demur, given the proviso (which is an initial condition of the problem):

(B) \[ p(\text{tails} \mid O \text{ contains } \$2n) = p(\text{heads} \mid O \text{ contains } \$2n) = 0.5. \]