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Results on the number of zeros in a disk for three types of polynomials

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Abstract. We impose a monotonicity condition with several reversals on the moduli of the coefficients of a polynomial. We then consider three types of polynomials: (1) those satisfying the condition on all of the coefficients, (2) those satisfying the condition on the even indexed and odd indexed coefficients separately, and (3) polynomials of the form $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ where $\mu \geq 1$ with the coefficients $a_\mu, a_{\mu+1}, \ldots, a_n$ satisfying the condition. For each type of polynomial, we give a result which puts a bound on the number of zeros in a disk (in the complex plane) centered at the origin. For each type, we give an example showing the results are best possible.

1. Introduction

A classical result in the study of the location of the zeros of a polynomial in the complex plane is the Eneström–Kakeya theorem.

Theorem 1.1 (Eneström–Kakeya theorem). *If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with real coefficients satisfying $0 \leq a_0 \leq a_1 \leq \cdots \leq a_n$, then all the zeros of P lie in $|z| \leq 1$.*

There is a huge number of generalizations of the Eneström–Kakeya theorem, most of which involve some sort of variant on the theme of monotonicity of the coefficients. For a recent survey of such results, see [4]. One example of such a result, which is particularly related to the results of this paper, is due to Chattopadhyay et al. [3].

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Theorem 1.2. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *where for some* $t > 0$ *and some* $0 = k_0 < k_1 < \cdots < k_r < k_{r+1} = n$ *we have*

$$0 < |a_0| \le t|a_1| \le t^2|a_2| \le \cdots \le t^{k_1}|a_{k_1}| \ge t^{k_1+1}|a_{k_1+1}| \ge \cdots$$

$$\ge t^{k_2}|a_{k_2}| \le t^{k_2+1}|a_{k_2+1}| \le \cdots \le t^{k_3}|a_{k_3}| \ge \cdots$$

(*with inequalities reversed at indices* $k_1, k_2, \ldots, k_r$ *and* $t^n|a_n|$ *is the last term in the inequality*). *Also suppose* $|arg(a_j) - \beta| \le \alpha \le \pi/2$ *for* $0 \le j \le n$ *and some real* α *and* β. *Then all zeros of* P *lie in* $R_1 \le |z| \le R_2$, *where* $R_1 = \min\{t|a_0|/M_1, t\}$, $R_2 = \max\{M_2/|a_0|, 1/t\}$,

$$M_1 = -\left\{2\cos\alpha \sum_{\ell=1}^{r}(-1)^{\ell}|a_{k_\ell}|t^{k_\ell} + |a_0| + (-1)^{\ell+1}|a_n|t^n\right\}$$
$$+ 2\sin\alpha \sum_{\ell=0}^{n-1}|a_j|t^j + (-|a_0| + |a_n|t^n)\sin\alpha + |a_n|t^n$$

and

$$M_2 = -\cos\alpha\left\{(t^2-1)\sum_{\ell=0}^{r}\left((-1)^{\ell+1}\sum_{s=k_\ell+1}^{k_{\ell+1}-1}|a_s|t^{n-s-1}\right)\right\}$$
$$-(t^2+1)\sum_{\ell=1}^{r}(-1)^{\ell}|a_{k_\ell}|t^{n-k_\ell-1} - |a_0|t^{n-1}(1+t^2) + (-1)^r|a_n|t$$
$$+\sin\alpha\left\{\sum_{j=1}^{n}(t|a_j| + |a_{j-1}|)t^{n-j}\right\} + |a_0|t^{n+1}.$$

Another related area of study is the *number* of zeros of a polynomial in a disk of a certain radius. For example, Pukhta [8] put a monotonicity condition on the moduli of the coefficients of a polynomial to prove the following theorem.

Theorem 1.3. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *where* $0 < |a_0| \le |a_1| \le \cdots \le |a_n|$. *Also suppose* $|arg(a_j) - \beta| \le \alpha \le \pi/2$ *for* $0 \le j \le n$ *and some real* α *and* β. *Then for* $0 < \delta < 1$ *the number of zeros of* $P(z)$ *in the disk* $|z| \le \delta$ *is less than*

$$\frac{1}{\log(1/\delta)}\log\left(\frac{|a_n|(\cos\alpha + \sin\alpha + 1) + 2\sin\alpha\sum_{j=0}^{n-1}|a_j|}{|a_0|}\right).$$

The purpose of this paper is to apply the r reversals hypothesis of Chattopadhyay, Das, Jain, and Konwar to three types of polynomials and to produce corresponding number of zeros results similar to Theorem 1.3. The first type involves those polynomials which satisfy the multiple reversals monotonicity condition on the moduli of the coefficients, as given in Theorem 1.2. The second type involves polynomials which satisfy the same condition on the moduli of its even indexed coefficients and on its odd indexed coefficients separately. The third type involves those polynomials for which there is a gap in the coefficients; that is, polynomials of the form $a_0 + \sum_{j=\mu}^{n} a_j z^j$ for some $1 \leq \mu \leq n$. The multiple reversals monotonicity condition is then imposed on the moduli of coefficients $a_\mu, a_{\mu+1}, \ldots, a_n$. These three types of polynomials are addressed in the following three sections, respectively.

2. Monotonicity condition on the moduli of all coefficients

First, we consider polynomials which satisfy the hypotheses of Theorem 1.2 and give a bound on the number of zeros in a disk.

Theorem 2.1. *Let* $P(z) = \displaystyle\sum_{j=0}^{n} a_j z^j$ *where for some* $t > 0$ *and some* $0 = k_0 < k_1 < \cdots < k_r < k_{r+1} = n$ *we have*

$$0 < |a_0| \leq t|a_1| \leq t^2|a_2| \leq \cdots \leq t^{k_1}|a_{k_1}| \geq t^{k_1+1}|a_{k_1+1}| \geq \cdots$$
$$\geq t^{k_2}|a_{k_2}| \leq t^{k_2+1}|a_{k_2+1}| \leq \cdots \leq t^{k_3}|a_{k_3}| \geq \cdots$$

(*with inequalities reversed at indices* $k_1, k_2, \ldots, k_r$ *and* $t^n|a_n|$ *is the last term in the inequality*). *Also suppose* $|arg(a_j) - \beta| \leq \alpha \leq \pi/2$ *for* $0 \leq j \leq n$ *and some real* α *and* β. *Then for* $0 < \delta < 1$ *the number of zeros of* $P(z)$ *in the disk* $|z| \leq \delta t$ *is less than*

$$\frac{1}{\log(1/\delta)} \log \frac{M}{|a_0|},$$

where

$$M = |a_0|t(1 - \cos\alpha - \sin\alpha) + 2\cos\alpha \sum_{\ell=1}^{r} (-1)^{\ell+1}|a_{k_\ell}|t^{k_\ell+1}$$
$$+ 2\sin\alpha \sum_{j=0}^{n-1} |a_j|t^{j+1} + |a_n|t^{n+1}(1 + \sin\alpha + (-1)^r \cos\alpha).$$

When $r = 1$, Theorem 2.1 reduces to Theorem 1 of [5], which in turn implies Theorem 1.3.

If the coefficients of polynomial P are real and nonnegative, then we can take $\alpha = 0$ in Theorem 2.1. If, in addition, we take $t = 1$, then we get the following corollary.

Corollary 2.2. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *where each* a_i *is real and nonnegative, and for some* $0 = k_0 < k_1 < \cdots < k_r < k_{r+1} = n$ *we have*

$$0 < a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_{k_1} \geq a_{k_1+1} \geq \cdots \geq a_{k_2} \leq a_{k_2+1} \leq \cdots \leq a_{k_3} \geq \cdots$$

(*with inequalities reversed at indices* $k_1, k_2, \ldots, k_r$ *and* a_n *is the last term in the inequality*). *Then for* $0 < \delta < 1$ *the number of zeros of* $P(z)$ *in the disk* $|z| \leq \delta$ *is less than*

$$\frac{1}{\log(1/\delta)} \log \frac{M}{a_0},$$

where $M = 2\sum_{\ell=1}^{r}(-1)^{\ell+1}a_{k_\ell} + a_n(1 + (-1)^r)$.

We now give an example showing that Theorem 2.1 is best possible in certain cases. That is, there exists a polynomial P and a δ, with $0 < \delta < 1$, where the number of zeros of P in $|z| \leq \delta t$ equals the number predicted by Theorem 2.1.

Example 2.3. Consider $P(z) = 1+10z+z^2+1.1z^3+0.1z^4$. The four roots of P are approximately -10.916, -0.1001, $0.0082 + 3.0131i$, and $0.0082 - 3.0131i$. We can apply Corollary 2.2 to P with $r = 3$, $a_{k_1} = 10$, $a_{k_2} = 1$, and $a_{k_3} = 1.1$. We get $M = 2(a_{k_1} - a_{k_2} + a_{k_3}) = 20.2$ and with $\delta = 0.20$, $\frac{1}{\log(1/\delta)} \log\left(\frac{M}{a_0}\right) = \frac{1}{\log(1/0.2)} \log\left(\frac{20.2}{1}\right) = 1.8675$. So Corollary 2.2 implies that P has at most 1 zero in $|z| \leq 0.20$ and, in fact, P has exactly 1 zero in this disk. This example shows that Theorem 2.1 is best possible in some cases.

The next example shows that it is possible to use the number of zeros results to actually locate all zeros of a polynomial. This can be done for a given n degree polynomial P which satisfies the hypotheses of the result by finding a value of δ such that the result predicts that n zeros of P lie in the given disk.

Example 2.4. Consider $P(z) = 1 + 2z + z^2 + 100z^3$. We can apply Corollary 2.2 to P with $r = 2$, $a_{k_1} = 2$, $a_{k_2} = 1$, and $a_n = 100$. We get $M = 2(a_{k_1} - a_{k_2} + a_n) = 202$ and with $\delta = 0.25$, $\frac{1}{\log(1/\delta)} \log\left(\frac{M}{a_0}\right) = \frac{1}{\log(1/0.25)} \log\left(\frac{202}{1}\right) = 3.829$. So Corollary 2.1 implies that P has at most 3 zeros in $|z| \leq 0.25$. Since P is of degree 3, this means that all of the zeros of P lie in this disk.

3. Monotonicity condition on the even indexed and odd indexed coefficients

Cao and Gardner [1] gave an Eneström–Kakeya style result by imposing a monotonicity condition on the even indexed and odd indexed coefficients of a polynomial separately.

Notice that for any $n \in \mathbb{N}$, we have:

(i) if n is even, then $2\lfloor n/2 \rfloor = n$ and $2\lfloor (n+1)/2 \rfloor - 1 = n-1$,

(ii) if n is odd, then $2\lfloor n/2 \rfloor = n-1$ and $2\lfloor (n+1)/2 \rfloor - 1 = n$.

In either case, $2\lfloor n/2 \rfloor$ is the largest even integer less than or equal to n and $2\lfloor (n+1)/2 \rfloor - 1$ is the largest odd integer less than or equal to n. We need this observation when imposing the monotonicity condition on the even indexed coefficients and odd indexed coefficients, as illustrated in the next result.

Theorem 3.1. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *where for some* $t > 0$, *for some* $0 = 2k_0^e < 2k_1^e < \cdots < 2k_{r_1}^e < 2k_{r_1+1}^e = 2\lfloor n/2 \rfloor$ *we have* [1]

$$0 < |a_0| \leq t^2|a_2| \leq t^4|a_4| \leq \cdots \leq t^{2k_1^e}|a_{2k_1^e}| \geq t^{2k_1^e+1}|a_{2k_1^e+1}| \geq \cdots$$
$$\geq t^{2k_2^e}|a_{2k_2^e}| \leq t^{2k_2^e+1}|a_{2k_2^e+1}| \leq \cdots \leq t^{2k_3^e}|a_{2k_3^e}| \geq \cdots$$

(*with inequalities reversed at indices* $2k_1^e, 2k_2^e, \ldots, 2k_{r_1}^e$ *and* $t^{2\lfloor n/2 \rfloor}|a_{2\lfloor n/2 \rfloor}|$ *is the last term in the inequality*), *and for some* $1 = 2k_0^o - 1 < 2k_1^o - 1 < \cdots < 2k_{r_2}^o - 1 < 2k_{r_2+1}^o - 1 = 2\lfloor (n+1)/2 \rfloor - 1$ *we have*

$$|a_1| \leq t^2|a_3| \leq t^4|a_5| \leq \cdots \leq t^{2k_1^o-2}|a_{2k_1^o-1}| \geq t^{2k_1^o}|a_{2k_1^o+1}| \geq \cdots$$
$$\geq t^{2k_2^o-2}|a_{2k_2^o-1}| \leq t^{2k_2^o}|a_{2k_2^o+1}| \leq \cdots \leq t^{2k_3^o-2}|a_{2k_3^o-1}| \geq \cdots$$

(*with inequalities reversed at indices* $2k_1^o - 1$, $2k_2^o - 1$, $\ldots$, $2k_{r_2}^o - 1$ *and* $t^{2\lfloor (n+1)/2 \rfloor - 2}|a_{2\lfloor (n+1)/2 \rfloor - 1}|$ *is the last term in the inequality*). *Also suppose* $|\arg(a_j) - \beta| \leq \alpha \leq \pi/2$ *for* $0 \leq j \leq n$ *and some real* α *and* β. *Then for* $0 < \delta < 1$ *the number of zeros of* $P(z)$ *in the disk* $|z| \leq \delta t$ *is less than*

$$\frac{1}{\log(1/\delta)} \log \frac{M}{|a_0|},$$

where

$$M = (|a_0|t^2 + |a_1|t^3)(1 - \cos\alpha + \sin\alpha) + (|a_{n-1}|t^{n+1} + |a_n|t^{n+2})(1 - \sin\alpha)$$
$$+|a_{2\lfloor n/2 \rfloor}|t^{2\lfloor n/2 \rfloor + 2}(-1)^{r_1}\cos\alpha + |a_{2\lfloor (n+1)/2 \rfloor - 1}|t^{2\lfloor (n+1)/2 \rfloor + 1}(-1)^{r_2}\cos\alpha$$
$$+2\sin\alpha \sum_{j=0}^{n} |a_j|t^{j+2} + 2\cos\alpha \left[\sum_{\ell=1}^{r_1}(-1)^{\ell+1}|a_{2k_\ell^e}|t^{2k_\ell^e+2}\right.$$
$$\left. + \sum_{\ell=1}^{r_2}(-1)^{\ell+1}|a_{2k_\ell^o-1}|t^{2k_\ell^o+1}\right].$$

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[1] We use a superscript of "e" on the indices k_i when dealing with even indexed coefficients and use a superscript of "o" when dealing with odd indices.

When $r = 1$, Theorem 3.1 reduces to Theorem 2.1 of [6].

If the coefficients of polynomial P are real and nonnegative, then we can take $\alpha = 0$ in Theorem 3.1. If, in addition, we take $t = 1$, then we get the following corollary.

Corollary 3.2. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ where each a_i is real and nonnegative, and for some $0 = 2k_0^e < 2k_1^e < \cdots < 2k_{r_1}^e < 2k_{r_1+1}^e = 2\lfloor n/2 \rfloor$ we have*

$$0 < a_0 \le a_2 \le a_4 \le \cdots \le a_{2k_1^e} \ge a_{2k_1^e+1} \ge \cdots \ge a_{2k_2^e}$$
$$\le a_{2k_2^e+1} \le \cdots \le a_{2k_3^e} \ge \cdots$$

(with inequalities reversed at indices $2k_1^e, 2k_2^e, \ldots, 2k_{r_1}^e$ and $a_{2\lfloor n/2 \rfloor}$ is the last term in the inequality), and for some $1 = 2k_0^o - 1 < 2k_1^o - 1 < \cdots < 2k_{r_2}^o - 1 < 2k_{r_2+1}^o - 1 = 2\lfloor (n+1)/2 \rfloor - 1$ we have

$$0 \le a_1 \le a_3 \le a_5 \le \cdots \le a_{2k_1^o-1} \ge a_{2k_1^o+1} \ge \cdots \ge a_{2k_2^o-1}$$
$$\le a_{2k_2^o+1} \le \cdots \le a_{2k_3^o-1} \ge \cdots$$

(with inequalities reversed at indices $2k_1^o - 1, 2k_2^o - 1, \ldots, 2k_{r_2}^o - 1$ and $a_{2\lfloor (n+1)/2 \rfloor - 1}$ is the last term in the inequality). Then for $0 < \delta < 1$ the number of zeros of $P(z)$ in the disk $|z| \le \delta$ is less than

$$\frac{1}{\log(1/\delta)} \log \frac{M}{a_0},$$

where

$$M = a_{n-1} + a_n + a_{2\lfloor n/2 \rfloor}(-1)^{r_1} + a_{2\lfloor (n+1)/2 \rfloor - 1}(-1)^{r_2}$$
$$+2\left[\sum_{\ell=1}^{r_1}(-1)^{\ell+1} a_{2k_\ell^e} + \sum_{\ell=1}^{r_2}(-1)^{\ell+1} a_{2k_\ell^o - 1}\right].$$

We now give an example showing that Theorem 3.1 is best possible in certain cases.

Example 3.3. Consider $P(z) = 1 + z + 2z^2 + 2z^3 + z^4 + z^5 + 1000z^6 + 2z^7$. The seven roots of P are approximately -499.999, $-0.2666 + 0.0151i$, $-0.2666 - 0.0151i$, $-0.0145 + 0.3073i$, $-0.0145 - 0.3073i$, $0.2806 + 0.1839i$, $0.2806 - 0.1839i$. We can apply Corollary 3.2 to P with $r_1 = 2$, $r_2 = 2$, $a_{2k_1^o-1} = 1$, $a_{2k_1^e} = 2$, $a_{2k_2^o-1} = 2$, and $a_{2k_2^e} = 1$. We get $M = a_{n-1} + a_n + a_{n-1} + a_n + 2[a_{2k_1^e} - a_{2k_2^e} + a_{2k_1^o-1} - a_{2k_2^o-1}] = 2008$, and with $\delta = 0.336$, $\frac{1}{\log(1/\delta)} \log\left(\frac{M}{a_0}\right) = \frac{1}{\log(1/0.336)} \log\left(\frac{2008}{1}\right) = 6.9728$. So Corollary 3.2 implies that P has at most 6 zeros in $|z| \le 0.336$ and, in fact, P has exactly 6 zeros in this disk. This example shows that Theorem 3.1 is best possible in some cases.

4. Monotonicity condition on the moduli of the coefficients for polynomials with a gap in the coefficients

Chan and Malik [2] introduced the class of polynomials of the form $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ where $\mu \geq 1$. They proved a Bernstein inequality (in particular, a generalization of the Erdös–Lax theorem) for this type polynomials. We now consider these polynomials and impose a monotonicity condition on the moduli of the coefficients in order to produce a number of zeros result.

Theorem 4.1. *Let $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ for some $1 \leq \mu \leq n$, where $a_0 \neq 0$ and for some $t > 0$ and some $1 \leq \mu = k_0 < k_1 < \cdots < k_r < k_{r+1} = n$ we have*

$$t^{\mu}|a_{\mu}| \leq t^{\mu+1}|a_{\mu+1}| \leq t^{\mu+2}|a_{\mu+2}| \leq \cdots \leq t^{k_1}|a_{k_1}| \geq t^{k_1+1}|a_{k_1+1}| \geq \cdots$$

$$\geq t^{k_2}|a_{k_2}| \leq t^{k_2+1}|a_{k_2+1}| \leq \cdots \leq t^{k_3}|a_{k_3}| \geq \cdots$$

(with inequalities reversed at indices $k_1, k_2, \ldots, k_r$ and $t^n|a_n|$ is the last term in the inequality). Also suppose $|arg(a_j) - \beta| \leq \alpha \leq \pi/2$ for $0 \leq j \leq n$ and some real α and β. Then for $0 < \delta < 1$ the number of zeros of $P(z)$ in the disk $|z| \leq \delta t$ is less than

$$\frac{1}{\log(1/\delta)} \log \frac{M}{|a_0|},$$

where

$$M = 2|a_0|t + |a_{\mu}|t^{\mu+1}(1 - \cos\alpha - \sin\alpha) + 2\cos\alpha \sum_{\ell=1}^{r} (-1)^{\ell+1}|a_{k_\ell}|t^{k_\ell+1}$$

$$+2\sin\alpha \sum_{j=\mu}^{n} |a_j|t^{j+1} + |a_n|t^{n+1}(1 + \sin\alpha + (-1)^r \cos\alpha).$$

If the coefficients of polynomial P are real and nonnegative, then we can take $\alpha = 0$ in Theorem 4.1. If, in addition, we take $t = 1$, then we get the following corollary.

Corollary 4.2. *Let $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ where each a_i is real and nonnegative, and for some $1 \leq \mu = k_0 < k_1 < \cdots < k_r < k_{r+1} = n$ we have*

$$0 < a_{\mu} \leq a_{\mu+1} \leq a_{\mu+2} \leq \cdots \leq a_{k_1} \geq a_{k_1+1} \geq \cdots \geq a_{k_2}$$

$$\leq a_{k_2+1} \leq \cdots \leq a_{k_3} \geq \cdots$$

(*with inequalities reversed at indices $k_1, k_2, \ldots, k_r$ and a_n is the last term in the inequality*). *Then for $0 < \delta < 1$ the number of zeros of $P(z)$ in the disk $|z| \le \delta$ is less than*

$$\frac{1}{\log(1/\delta)} \log \frac{M}{|a_0|},$$

where

$$M = 2a_0 + 2\sum_{\ell=1}^{r}(-1)^{\ell+1}a_{k_\ell} + a_n t^{n+1}(1 + (-1)^r).$$

We now give an example showing that Theorem 3.1 is best possible in certain cases.

Example 4.3. Consider $P(z) = 1+8z^2+8z^3+z^4+1.1z^5+0.1z^6$. The six roots of P are approximately -10.6988, -1.0925, $0.0526+0.3347i$, $0.0526-0.3347i$, $0.3430+2.7088i$, and $0.3430-2.7088i$. We can apply Corollary 4.2 to P with $r=3$, $a_{k_1}=8$, $a_{k_2}=1$, and $a_{k_3}=1.1$. We get $M = 2a_0+2(a_{k_1}-a_{k_2}+a_{k_3}) = 18.2$ and with $\delta = 0.35$, $\frac{1}{\log(1/\delta)}\log\left(\frac{M}{a_0}\right) = \frac{1}{\log(1/0.35)}\log\left(\frac{18.2}{1}\right) = 2.7637$. So Corollary 4.2 implies that P has at most 2 zeros in $|z| \le 0.35$ and, in fact, P has exactly 2 zeros in this disk. This example shows that Theorem 4.1 is best possible in some cases.

5. Proofs of theorems

We need as a lemma a result which appears in Titchmarsh's book [9], page 171.

Lemma 5.1. *Let $F(z)$ be analytic in $|z| \le R$. Let $|F(z)| \le M$ in the disk $|z| \le R$ and suppose $F(0) \ne 0$. Then for $0 < \delta < 1$ the number of zeros of $F(z)$ in the disk $|z| \le \delta R$ is less than*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|F(0)|}.$$

The following is due to Govil and Rahman and appears in [7].

Lemma 5.2. *Let $z, z' \in \mathbb{C}$ with $|z| \ge |z'|$. Suppose $|arg(z^* - \beta)| \le \alpha \le \pi/2$ for $z^* \in \{z, z'\}$ and for some real α and β. Then*

$$|z - z'| \le (|z| - |z'|)\cos\alpha + (|z| + |z'|)\sin\alpha.$$

We now give proofs of our three main theorems.

Proof of Theorem 2.1. Consider

$$F(z) = (t-z)P(z) = (t-z)\sum_{j=0}^{n} a_j z^j = \sum_{j=0}^{n}(a_j t z^j - a_j z^{j+1})$$

$$= a_0 t + \sum_{j=1}^{n} a_j t z^j - \sum_{j=1}^{n} a_{j-1} z^j - a_n z^{n+1} = a_0 t + \sum_{j=1}^{n} (a_j t - a_{j-1}) z^j - a_n z^{n+1}.$$

For $|z| = t$ we have

$$\begin{aligned}
|F(z)| &\leq |a_0|t + \sum_{j=1}^{n} |a_j t - a_{j-1}| t^j + |a_n| t^{n+1} \\
&= |a_0|t + \sum_{j=1}^{k_1} |a_j t - a_{j-1}| t^j + \sum_{j=k_1+1}^{k_2} |a_{j-1} - a_j t| t^j \\
&\quad + \sum_{j=k_2+1}^{k_3} |a_{j-1} - a_j t| t^j + \sum_{j=k_3+1}^{k_4} |a_{j-1} - a_j t| t^j + \\
&\quad \cdots + \sum_{j=k_{r-2}+1}^{k_{r-1}} |a_{j-1} - a_j t| t^j + \sum_{j=k_{r-1}+1}^{k_r} |a_{j-1} - a_j t| t^j \\
&\quad + \sum_{j=k_r+1}^{n} |a_{j-1} - a_j t| t^j + |a_n| t^{n+1} \\
&= |a_0|t + \sum_{\ell=0}^{r} \left(\sum_{j=k_\ell+1}^{k_{\ell+1}} |a_j + a_{j-1}| t^j \right) + |a_n| t^{n+1} \\
&\leq |a_0|t + \sum_{\substack{\ell=0 \\ \ell \text{ even}}}^{r} \left(\sum_{j=k_\ell+1}^{k_{\ell+1}} \{(|a_j|t - |a_{j-1}|) \cos\alpha \right. \\
&\quad \left. + (|a_{j-1}| + |a_j|t) \sin\alpha\} t^j \right) + \sum_{\substack{\ell=0 \\ \ell \text{ odd}}}^{r} \left(\sum_{j=k_\ell+1}^{k_{\ell+1}} \{(|a_{j-1}| - |a_j|t) \cos\alpha \right. \\
&\quad \left. + (|a_j|t + |a_{j-1}|) \sin\alpha\} t^j \right) + |a_n| t^{n+1} \\
&\quad \text{by Lemma 5.2 with } z = a_j t \text{ and } z' = a_{j-1} \text{ for } \ell \text{ even;} \\
&\quad \qquad\qquad \text{and } z = a_{j-1} \text{ and } z' = a_j t \text{ for } \ell \text{ odd} \\
&= |a_0|t + \sum_{\substack{\ell=0 \\ \ell \text{ even}}}^{r} (-1)^\ell \left[-|a_{k_\ell}| t^{k_\ell+1} + |a_{k_{\ell+1}}| t^{k_{\ell+1}+1} \right] \cos\alpha \\
&\quad + \sum_{\substack{\ell=0 \\ \ell \text{ odd}}}^{r} (-1)^\ell \left[-|a_{k_\ell}| t^{k_\ell+1} + |a_{k_{\ell+1}}| t^{k_{\ell+1}+1} \right] \cos\alpha
\end{aligned}$$

$$
\begin{aligned}
&+|a_0|t\sin\alpha + 2\sum_{j=1}^{n-1}|a_j|t^{j+1}\sin\alpha + |a_n|t^{n+1}\sin\alpha + |a_n|t^{n+1} \\
= \;& |a_0|t(1+\sin\alpha) + \left[-|a_{k_0}|t^{k_0+1} + |a_{k_1}|t^{k_1+1}\right]\cos\alpha + \left[|a_{k_1}|t^{k_1+1}\right. \\
&\left.-|a_{k_2}|t^{k_2+1}\right]\cos\alpha + \cdots + (-1)^r\left[-|a_{k_r}|t^{k_r+1} + |a_{k_{r+1}}|t^{k_{r+1}+1}\right]\cos\alpha \\
&+2\sum_{j=1}^{n-1}|a_j|t^{j+1}\sin\alpha + |a_n|t^{n+1}(1+\sin\alpha) \\
= \;& |a_0|t(1-\cos\alpha-\sin\alpha) + 2\cos\alpha\sum_{\ell=1}^{r}(-1)^{\ell+1}|a_{k_\ell}|t^{k_\ell+1} \\
&+2\sin\alpha\sum_{j=0}^{n-1}|a_j|t^{j+1} + |a_n|t^{n+1}(1+\sin\alpha+(-1)^r\cos\alpha) = M.
\end{aligned}
$$

Now $F(z)$ is analytic in $|z| \le t$, and $|F(z)| \le M$ for $|z| = t$. So by Lemma 5.1 and the Maximum Modulus Theorem, the number of zeros of F (and hence of P) in $|z| \le \delta t$ is less than or equal to

$$\frac{1}{\log 1/\delta}\log\frac{M}{|a_0|}.$$

The result now follows. □

Proof of Theorem 3.1. Consider

$$G(z) = (t^2 - z^2)P(z) = (t^2 - z^2)\sum_{j=0}^{n} a_j z^j$$

$$= a_0t^2 + a_1t^2z + \sum_{j=2}^{n}(a_jt^2 - a_{j-2})z^j - a_{n-1}z^{n+1} - a_nz^{n+2}.$$

For $|z| = t$ we have

$$
\begin{aligned}
|G(z)| \;\le\;& |a_0|t^2 + |a_1|t^3 + \sum_{j=2}^{n}|a_jt^2 - a_{j-2}t|t^j + |a_{n-1}|t^{n+1} + |a_n|t^{n+2} \\
= \;& |a_0|t^2 + |a_1|t^3 + \sum_{\substack{j=2 \\ j \text{ even}}}^{2\lfloor n/2\rfloor}|a_jt^2 - a_{j-2}|t^j + \sum_{\substack{j=3 \\ j \text{ odd}}}^{2\lfloor (n+1)/2\rfloor-1}|a_jt^2 - a_{j-2}|t^j \\
&+|a_{n-1}|t^{n+1} + |a_n|t^{n+2}
\end{aligned}
$$

$$
\begin{aligned}
&= |a_0|t^2 + |a_1|t^3 + \sum_{\ell=0}^{r_1}\left(\sum_{\substack{j=2k_\ell^e+2\\ j \text{ even}}}^{2k_{\ell+1}^e} |a_j t^2 - a_{j-2}|t^j\right)\\
&\quad + \sum_{\ell=0}^{r_2}\left(\sum_{\substack{j=2k_\ell^o+1\\ j \text{ odd}}}^{2k_{\ell+1}^o-1} |a_j t^2 - a_{j-2}|t^j\right) + |a_{n-1}|t^{n+1} + |a_n|t^{n+2}\\
&\le |a_0|t^2 + |a_1|t^3\\
&\quad + \sum_{\substack{\ell=0\\ \ell \text{ even}}}^{r_1}\left(\sum_{\substack{j=2k_\ell^e+2\\ j \text{ even}}}^{2k_{\ell+1}^e} \left\{(|a_j|t^2 - |a_{j-2}|)\cos\alpha + (|a_{j-2}| + |a_j|t^2)\sin\alpha\right\} t^j\right)\\
&\quad + \sum_{\substack{\ell=1\\ \ell \text{ odd}}}^{r_1}\left(\sum_{\substack{j=2k_\ell^e+2\\ j \text{ even}}}^{2k_{\ell+1}^e} \left\{(|a_{j-2}| - |a_j|t^2)\cos\alpha + (|a_{j-2}| + |a_j|t^2)\sin\alpha\right\} t^j\right)\\
&\quad + \sum_{\substack{\ell=0\\ \ell \text{ even}}}^{r_2}\left(\sum_{\substack{j=2k_\ell^o+1\\ j \text{ odd}}}^{2k_{\ell+1}^o-1} \left\{(|a_j|t^2 - |a_{j-2}|)\cos\alpha + (|a_{j-2}| + |a_j|t^2)\sin\alpha\right\} t^j\right)\\
&\quad + \sum_{\substack{\ell=1\\ \ell \text{ odd}}}^{r_2}\left(\sum_{\substack{j=2k_\ell^o+1\\ j \text{ odd}}}^{2k_{\ell+1}^o-1} \left\{(|a_{j-2}| - |a_j|t^2)\cos\alpha + (|a_{j-2}| + |a_j|t^2)\sin\alpha\right\} t^j\right)\\
&\quad +|a_{n-1}|t^{n+1} + |a_n|t^{n+2}\\
&\quad \text{by Lemma 5.2 with } z = a_j t \text{ and } z' = a_{j-2} \text{ for } \ell \text{ even;}\\
&\qquad\qquad\qquad \text{and } z = a_{j-2} \text{ and } z' = a_j t \text{ for } \ell \text{ odd}\\
&= |a_0|t^2(1-\cos\alpha+\sin\alpha) + |a_1|t^3(1-\cos\alpha+\sin\alpha)\\
&\quad +2\sum_{\ell=1}^{r_1}(-1)^{\ell+1}|a_{2k_\ell^e}|t^{2k_\ell^e+2}\cos\alpha + 2\sum_{\ell=1}^{r_1}|a_{2k_\ell^e}|t^{2k_\ell^e+2}\sin\alpha\\
&\quad +2\sin\alpha\sum_{\ell=0}^{r_1}\left(\sum_{\substack{j=2k_\ell^e+2\\ j \text{ even}}}^{2k_{\ell+1}^e-2}|a_j|t^{j+2}\right) + 2\sum_{\ell=1}^{r_2}(-1)^{\ell+1}|a_{2k_\ell^o-1}|t^{2k_\ell^o+1}\cos\alpha
\end{aligned}
$$

$$
\begin{aligned}
& +2\sum_{\ell=1}^{r_2}|a_{2k_\ell^o-1}|t^{2k_\ell^o+1}\sin\alpha+2\sin\alpha\sum_{\ell=0}^{r_2}\left(\sum_{\substack{j=2k_\ell^o+1\\ j\text{ odd}}}^{2k_{\ell+1}^o-3}|a_j|t^{j+2}\right)\\
& +|a_{n-1}|t^{n+1}(1+\sin\alpha)+|a_n|t^{n+1}(1+\sin\alpha)\\
& +|a_{2\lfloor n/2\rfloor}|t^{2\lfloor n/2\rfloor+2}(-1)^{r_1}\cos\alpha\\
& +|a_{2\lfloor (n+1)/2\rfloor-1}|t^{2\lfloor (n+1)/2\rfloor+1}(-1)^{r_2}\cos\alpha\\
= \;& |a_0|t^2(1-\cos\alpha+\sin\alpha)+|a_1|t^3(1-\cos\alpha+\sin\alpha)\\
& +|a_{2\lfloor n/2\rfloor}|t^{2\lfloor n/2\rfloor+2}(-1)^{r_1}\cos\alpha\\
& +|a_{2\lfloor (n+1)/2\rfloor-1}|t^{2\lfloor (n+1)/2\rfloor+1}(-1)^{r_2}\cos\alpha\\
& +|a_{n-1}|t^{n+1}(1+\sin\alpha)+|a_n|t^{n+2}(1+\sin\alpha)\\
& +2\sin\alpha\left(\sum_{\substack{j=2\\ j\text{ even}}}^{2\lfloor n/2\rfloor-2}|a_j|t^{j+2}\right)+2\sin\alpha\left(\sum_{\substack{j=3\\ j\text{ odd}}}^{2\lfloor (n+1)/2\rfloor-3}|a_j|t^{j+2}\right)\\
& +2\cos\alpha\left[\sum_{\ell=1}^{r-1}(-1)^{\ell+1}|a_{2k_\ell^e}|t^{2k_\ell^e+2}+\sum_{\ell=1}^{r_2}(-1)^{\ell+1}|a_{2k_\ell^o-1}|r^{2k_\ell^o+1}\right]\\
= \;& (|a_0|t^2+|a_1|t^3)(1-\cos\alpha+\sin\alpha)+(|a_{n-1}|t^{n+1}+|a_n|t^{n+2})(1-\sin\alpha)\\
& +|a_{2\lfloor n/2\rfloor}|t^{2\lfloor n/2\rfloor+2}(-1)^{r_1}\cos\alpha\\
& +|a_{2\lfloor (n+1)/2\rfloor-1}|t^{2\lfloor (n+1)/2\rfloor+1}(-1)^{r_2}\cos\alpha\\
& +2\sin\alpha\sum_{j=0}^{n}|a_j|t^{j+2}+2\cos\alpha\left[\sum_{\ell=1}^{r_1}(-1)^{\ell+1}|a_{2k_\ell^e}|t^{2k_\ell^e+2}\right.\\
& \left.+\sum_{\ell=1}^{r_2}(-1)^{\ell+1}|a_{2k_\ell^o-1}|t^{2k_\ell^o+1}\right]=M.
\end{aligned}
$$

The result now follows as in the proof of Theorem 2.1. ☐

Proof of Theorem 4.1. Consider

$$F(z)=(t-z)P(z)=(t-z)\left(a_0+\sum_{j=\mu}^{n}a_jz^j\right)=a_0(t-z)+\sum_{j=\mu}^{n}(a_jtz^j-a_jz^{j+1})$$

$$=a_0(t-z)+\sum_{j=\mu}^{n}a_jtz^j-\sum_{j=\mu+1}^{n+1}a_{j-1}z^j$$

$$= a_0(t-z) + a_\mu t z^\mu + \sum_{j=\mu+1}^{n} (a_j t - a_{j-1}) z^j - a_n z^{n+1}.$$

For $|z| = t$ we have

$$\begin{aligned}
|F(z)| &\le 2|a_0|t + |a_\mu| t^{\mu+1} + \sum_{j=\mu+1}^{n} |a_j t - a_{j-1}| t^j + |a_n| t^{n+1} \\
&= 2|a_0|t + |a_\mu| t^{\mu+1} + \sum_{j=\mu+1}^{k_1} |a_j t - a_{j-1}| t^j + \sum_{j=k_1+1}^{k_2} |a_{j-1} - a_j t| t^j \\
&\quad + \sum_{j=k_2+1}^{k_3} |a_j t - a_{j-1}| t^j + \sum_{j=k_3+1}^{k_4} |a_{j-1} - a_j t| t^j \\
&\quad + \cdots + \sum_{j=k_{r-2}+1}^{k_{r-1}} |a_j t - a_{j-1}| t^j + \sum_{j=k_{r-1}+1}^{k_r} |a_j t - a_{j-1}| t^j \\
&\quad + \sum_{j=k_r+1}^{n} |a_j t - a_{j-1}| t^j + |a_n| t^{n+1} \\
&= 2|a_0|t + |a_\mu| t^{\mu+1} + \sum_{\ell=0}^{r} \left(\sum_{j=k_\ell+1}^{k_{\ell+1}} |a_j t - a_{j-1}| t^j \right) + |a_n| t^{n+1} \\
&\le 2|a_0|t + |a_\mu| t^{\mu+1} + \sum_{\substack{\ell=0 \\ \ell \text{ even}}}^{r} \left(\sum_{j=k_\ell+1}^{k_{\ell+1}} \{(|a_j|t - |a_{j-1}|)\cos\alpha \right. \\
&\quad \left. + (|a_{j-1}| + |a_j|t)\sin\alpha\} t^j \right) \\
&\quad + \sum_{\substack{\ell=0 \\ \ell \text{ odd}}}^{r} \left(\sum_{j=k_\ell+1}^{k_{\ell+1}} \{(|a_{j-1}| - |a_j|t)\cos\alpha + (|a_j|t + |a_{j-1}|)\sin\alpha\} t^j \right) \\
&\quad + |a_n| t^{n+1} \\
&\quad \text{by Lemma 5.2 with } z = a_j t \text{ and } z' = a_{j-1} \text{ for } \ell \text{ even;} \\
&\quad \qquad \text{and } z = a_{j-1} \text{ and } z' = a_j t \text{ for } \ell \text{ odd} \\
&= 2|a_0|t + |a_\mu| t^{\mu+1}(1+\sin\alpha) \\
&\quad + \sum_{\substack{\ell=0 \\ \ell \text{ even}}}^{r} (-1)^\ell \left[-|a_{k_\ell}| t^{k_\ell+1} + |a_{k_{\ell+1}}| t^{k_{\ell+1}+1} \right] \cos\alpha
\end{aligned}$$

$$
\begin{aligned}
& + \sum_{\substack{\ell=0 \\ \ell \text{ odd}}}^{r} (-1)^{\ell} \left[-|a_{k_\ell}| t^{k_\ell+1} + |a_{k_{\ell+1}}| t^{k_{\ell+1}+1} \right] \cos\alpha \\
& + |a_\mu| t^{\mu+1} \sin\alpha + 2 \sum_{j=\mu+1}^{n-1} |a_j| t^{j+1} \sin\alpha + |a_n| t^{n+1} \sin\alpha + |a_n| t^{n+1} \\
= \; & 2|a_0| t + |a_\mu| t^{\mu+1} (1 + \sin\alpha) \\
& + \sum_{\ell=0}^{r} (-1)^{\ell} \left[-|a_{k_\ell}| t^{k_\ell+1} + |a_{k_{\ell+1}}| t^{k_{\ell+1}+1} \right] \cos\alpha \\
& + 2 \sum_{j=\mu+1}^{n-1} |a_j| t^{j+1} \sin\alpha + |a_n| t^{n+1} (1 + \sin\alpha) \\
= \; & 2|a_0| t + |a_\mu| t^{\mu+1} (1 - \cos\alpha - \sin\alpha) + 2 \sum_{\ell=1}^{r} (-1)^{\ell+1} |a_{k_\ell}| t^{k_\ell+1} \cos\alpha \\
& + 2 \sum_{j=\mu}^{n-1} |a_j| t^{j+1} \sin\alpha + |a_n| t^{n+1} (1 + \sin\alpha + (-1)^r \cos\alpha).
\end{aligned}
$$

The result now follows as in the proof of Theorem 2.1. □

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