# An Alternate Approach to the Measure of a Set of Real Numbers

**Seminar on the History and Exploration of Math Problems (S.H.E.M.P.)**

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# 1. INTRODUCTION

**Note.** In these notes, we "compare and contrast" the approach to Lebesgue measure taken in H.L. Royden and P.M. Fitzpatrick's *Real Analysis* (4th Edition), Prentice Hall (2010) to the approach taken in A.M. Bruckner, J.B. Bruckner, and B.S. Thomson's *Real Analysis*, Prentice Hall (1997). ETSU's Real Analysis 1 (MATH 5210) uses the Royden and Fitzpatrick text and defines a set to be Lebesgue measurable if it satisfies the Carathéodory splitting condition. Bruckner, Bruckner, and Thomson define inner and outer measure and define a set to be Lebesgue *measurable* if its inner measure equals its outer measure. Henri Lebesgue himself used inner and outer measure in his foundational work of 1902. It was several years later that the Carathéodory splitting condition followed (in 1914). We show in these notes that the two approaches are equivalent.

**Note.** In graduate Real Analysis 1 (MATH 5210), we follow Royden and Fitzpatrick's definition of *outer measure* of a set of real numbers $E$ as

$$\mu^*(E) = \inf\left\{ \sum_{k=1}^{\infty} \ell(I_k) \,\middle|\, E \subset \cup_{k=1}^{\infty} I_k, \text{ and each } I_k \text{ is an open interval} \right\},$$

where $\ell(I)$ denotes the length of interval $I$. In this way, the outer measure of every set is defined since every set off nonnegative real numbers has an infimum. HEY, it's part of the definition of $\mathbb{R}$!

**Note.** We can then show that $\mu^*$ is:

**(1)** translation invariant ($\mu^*(E+x) = \mu^*(E)$ for all $x \in \mathbb{R}$),

**(2)** monotone ($A \subset B$ implies $\mu^*(A) \leq \mu^*(B)$),

**(3)** the outer measure of an interval is its length: $\mu^*(I) = \ell(I)$ for all intervals $I \subset \mathbb{R}$, and

**(4)** countably subadditive $\mu^*\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} \mu^*(E_k)$.

**Note.** We want countable additivity:

$$\mu^*\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu^*(E_k) \text{ when } E_i \cap E_j = \emptyset \text{ for } i \neq j.$$

To this end, the *Carathéodory Condition* or the *splitting condition* on set $A$ is introduced:

$$\mu^*(X) = \mu^*(X \cap A) + \mu^*(X \setminus A) \text{ for all } X \subset \mathbb{R}.$$

Royden and Fitzpatrick then defines a set $A$ to be *Lebesgue measurable* if it satisfies the splitting condition and defines its *Lebesgue measure* as $\mu^*(A)$.

**Note.** It can be shown that Lebesgue measure is countably additive and that the Lebesgue measurable sets $\mathcal{M}$ form a $\sigma$-algebra (i.e., a collection of sets closed under countable unions and complements). Hence, the Borel sets (the $\sigma$-algebra generated by open intervals) are measurable.

**Note.** However, it is unclear as to why the splitting condition is the desired condition to yield the property of measurability. In this presentation, we give an alternate definition of measurability which is more natural but, ultimately, equivalent to the definition of Royden and Fitzpatrick.

# 2. $F_\sigma$ and $G_\delta$ Sets

**Note.** Recall that a set of real numbers is open if and only if it is a countable disjoint union of open intervals. Inspired by this result, we classify other types of sets which can be described in terms of open and closed sets.

**Definition.** A set $\mathcal{A}$ of subsets of some point set $X$ is a *$\sigma$-algebra* (or a *Borel field*) if

**(1)** if $A_1, A_2, A_3, \cdots \in \mathcal{A}$ then $\cup_{i=1}^{\infty} A_i \in \mathcal{A}$,

**(2)** if $A \in \mathcal{A}$ then $X \setminus A = A^c \in \mathcal{A}$, and

**(3)** if $A_1, A_2, A_3, \cdots \in \mathcal{A}$ then $\cap_{i=1}^{\infty} A_i \in \mathcal{A}$.

**Note.** By DeMorgan's Laws, part (3) of the above definition is redundant. We can simplify the definition to: "A $\sigma$-algebra is a collection of sets closed under countable unions and complements."

**Theorem 2.1.** Given any collection of sets $\mathcal{C}$ of subsets of point set $X$, there is a smallest $\sigma$-algebra that contains $\mathcal{C}$. That is, there is a $\sigma$-algebra $\mathcal{A}$ containing $\mathcal{C}$ such that if $\mathcal{B}$ is any $\sigma$-algebra containing $\mathcal{C}$, then $\mathcal{A} \subset \mathcal{B}$.

**Proof.** This is Proposition 1.13 in Royden and Fitzpatrick. The construction of $\mathcal{A}$ involves intersecting all algebras which contains $\mathcal{C}$. ∎

**Definition.** The collection $\mathcal{B}$ of *Borel sets* is the smallest $\sigma$-algebra containing which contains all of the open sets.

**Note.** The idea of generating a $\sigma$-algebra is used in Royden and Fitzpatrick's study of Lebesgue measurable sets and follows this outline:

**(1)** The open sets are measurable,

**(2)** If $A_1, A_2, A_3, \cdots$ are measurable, then $\cup_{i=1}^{\infty} A_i$ is measurable, and

**(3)** If $A$ is measurable, then $A^c$ is measurable.

(2) and (3) together imply that the measurable sets form a $\sigma$-algebra. (1) then implies that the Borel sets are measurable.

**Note.** Part of *my* interest lies in trying to find out what a set of real numbers "looks like." For example, we know that an open set of real numbers is a countable, disjoint union of open intervals. To this end, we define certain classes of sets. Think of the open sets as a starting point. We know that a *countable union* of open sets is open and a *finite intersection* of open sets is open. So to create a new collection of sets based on open sets, we could explore what results from a countable intersection of open sets (and, similarly, countable unions of closed sets).

**Definition.** A set which is a countable intersection of open sets is a *$G_\delta$-set.* A set which is a countable union of closed sets is an *$F_\sigma$-set.*

**Note.** One explanation for the above notation, is the following (this is Wikipedia's current [3/12/2016] story). In German, $G$ if for *Gebiet* ("area") and $\delta$ is for *Durchschnitt* ("intersection"). In French, $F$ is for *fermé* ("closed") and $\sigma$ is for *somme* ("union").

**Note.** By DeMorgan's Laws, we see that the complement of a $G_\delta$-set is an $F_\sigma$-set (and conversely). Since every open interval is a countable union of closed sets

$$(a,b) = \bigcup_{n=1}^{\infty} \left[a + \frac{1}{n}, b - \frac{1}{n}\right],$$

we see that every open set is $F_\sigma$ (and so every closed set is $G_\delta$).

**Notice.** We can, in a sense, say what a $G_\delta$-set "looks like"—it is a countable intersection of countable unions of open intervals!

**Note.** Next, we introduce another "layer" of sets by considering countable unions and intersections again.

**Definition.** A set which is a countable union of $G_\delta$ sets is a $G_{\delta\sigma}$-set. A set which is a countable intersection of $F_\sigma$-sets is an $F_{\sigma\delta}$-set.

**Note.** Continuing in this fashion, alternating countable intersections and countable unions, we generate the following classes of sets:

$$G_\delta, G_{\delta\sigma}, G_{\delta\sigma\delta}, G_{\delta\sigma\delta\sigma}, \ldots$$

$$F_\sigma, F_{\sigma\delta}, F_{\sigma\delta\sigma}, F_{\sigma\delta\sigma\delta}, \ldots$$

Since the "$G$ chain" is based on open sets and the "$F$ chain" is based on closed sets, we see that all of these types of sets are in the $\sigma$-algebra generated by the open sets—that is, they are all Borel sets.

**Note.** Tangible applications of some of the "low order" Borel sets include the following two problems from Royden and Fitzpatrick:

**1.56.** Let $f$ be a real-valued function defined for all real numbers. Then the set of points at which $f$ is continuous is a $G_\delta$-set.

**1.57.** Let $\langle f_n \rangle$ be a sequence of continuous functions defined on $\mathbb{R}$. Then the set $C$ of points where this sequence converges is an $F_{\sigma\delta}$ set and the set of points where this sequence diverges is a $G_{\delta\sigma}$-set.

**Note.** One can show (I am not that one... yet!) that there are Borel sets which are neither in the $G$ chain nor in the $F$ chain. So, although we know what the $G$ chain sets and the $F$ chain sets look like, we still don't have a grasp on what general the Borel sets look like!

**Note.** We adopt a notation consistent with the assumption of the Continuum Hypothesis: $|\mathcal{P}(\mathbb{R})| = \aleph_2$. According to Corollary 4.5.3 of Iner Rana's *An Introduction to Measure and Integration* (2nd Edition, A.M.S. Graduate Studies in Mathematics, Volume 45, 2002), the cardinality of the Borel sets is $|\mathcal{B}| = c$, the cardinality of the continuum. But then, under the Continuum Hypothesis, $|\mathcal{B}| = \aleph_1$. So, with regard to $\mathcal{P}(\mathbb{R})$, "very few" sets are Borel sets.

**Note.** We see in Real Analysis 1 that there are $|\mathcal{P}(\mathbb{R})| = \aleph_2$ measurable sets (in fact, we can take the power set of the Cantor set and see that there are this many sets of measure 0). There are also $|\mathcal{R}(\mathbb{R})| = \aleph_2$ nonmeasurable sets (by taking a nonmeasurable set from $[1, 2]$ and unioning it with each of the measurable subsets of the Cantor set). Therefore "very few" measurable sets are Borel sets.

# 3. Outer and Inner Measure

**Note.** The following notes, definitions, and notation are based largely on *Real Analysis* by A.M. Bruckner, J.B. Bruckner, and B.S. Thomson, Prentice Hall 1997.

**Definition.** For any open interval $I = (a, b)$, define $\lambda(I) = b - a$.

**Recall.** A set of real numbers $G$ is open if and only if it is a countable disjoint union of open intervals:

$$G = \cup_{k=1}^{\infty} I_k \text{ where } I_j \cap I_k = \emptyset \text{ if } j \neq k$$

where each $I_k$ is an open interval.

**Definition.** For the above open set of real numbers $G = \cup_{k=1}^{\infty} I_k$ define

$$\lambda(G) = \sum_{k=1}^{\infty} \lambda(I_k).$$

If one of the $I_k$ is unbounded, define $\lambda(G) = \infty$ and if $G = \emptyset$ define $\lambda(G) = 0$.

**Definition.** Let $E$ be a bounded closed set with $a = \text{glb}(E)$ and $b = \text{lub}(E)$ (that is, $[a, b]$ is the smallest closed interval containing $E$). Define

$$\lambda(E) = b - a - \lambda((a, b) \setminus E).$$

**Notice.** If $E$ is closed, then $(a, b) \setminus E = (a, b) \cap E^c$ is open. Also, we get by rearranging:

$$\lambda(E) + \lambda((a, b) \setminus E) = b - a.$$

**Note.** We have $\lambda$ defined on any open set or any closed and bounded set. We now use $\lambda$ defined on the open sets to define outer measure, identical to Royden and Fitzpatrick's approach.

**Definition.** Let $E$ be an arbitrary subset of $\mathbb{R}$. Let

$$\lambda^*(E) = \inf\{\lambda(g) \mid E \subset G, G \text{ is open}\}.$$

Then $\lambda^*(E)$ is called the Lebesgue *outer measure* of $E$.

**Note.** By definition, for open $G$, $\lambda^*(G) = \lambda(G)$.

**Theorem 3.1.** For every $E \subset \mathbb{R}$, there exists a $G_\delta$ set $G$ such that $E \subset G$ and $\lambda^*(E) = \lambda^*(G)$. $G$ is called a *measurable cover* for $E$.

**Proof.** This is a result in Royden and Fitzpatrick (Theorem 2.11(ii)). ∎

**Note.** Since for open $G$ (with the notation from above), $\lambda(G) = \sum_{k=1}^{\infty} \lambda(I_k)$, we immediately have

$$\lambda^*(E) = \inf\left\{\sum_{k=1}^{\infty} \lambda(I_k) \mid E \subset \bigcup_{k=1}^{\infty} I_k, \text{ each } I_k \text{ an open interval}\right\}.$$

This is the same as Royden and Fitzpatrick's definition of outer measure $\mu^*$. As previously mentioned, we show in Real Analysis 1 that $\lambda^* = \mu^*$ is (1) translation invariant, (2) monotone, (3) the outer measure of an interval is its length, and (4) countably subadditive.

**Note.** It would seem that $\lambda^*$ should do for a measure. However, $\lambda^*$ is not countably additive. In fact, there are disjoint sets $E_1$ and $E_2$ such that

$$\lambda(E_1 \cup E_2) = \lambda^*(E_1) + \lambda^*(E_2)$$

*does not* hold. Specific examples of such sets are seen with the construction of a nonmeasurable set (climaxing in the "offensive" Banach-Tarski Paradox).

**Definition.** Let $E$ be an arbitrary subset of $\mathbb{R}$. Let

$$\lambda_*(E) = \sup\{\lambda(F) \mid F \subset E, F \text{ is compact}\}.$$

Then $\lambda_*(E)$ is called the Lebesgue *inner measure* of $E$.

**Note.** By definition, for compact $F$, $\lambda_*(F) = \lambda(F)$.

**Note.** Similar to the proofs for $\mu^*$, we can show that $\lambda_*$ is:

**(1)** translation invariant ($\lambda_*(E + x) = \lambda_*(E)$ for all $x \in \mathbb{R}$),

**(2)** monotone ($A \subset B$ implies $\lambda_*(A) \leq \lambda_*(B)$),

**(3)** the inner measure of an interval is its length: $\lambda_*(I) = \ell(I)$ for all intervals $I \subset \mathbb{R}$, and

**(4)** countably superadditive

$$\lambda_*\left(\bigcup_{k=1}^{\infty} E_k\right) \geq \sum_{k=1}^{\infty} \lambda_*(E_k).$$

**Theorem 3.2.** For every $E \subset \mathbb{R}$, there exists an $F_\sigma$ set $F$ such that $F \subset E$ and $\lambda_*(F) = \lambda_*(E)$. $F$ is called a *measurable kernal* of $g$.

**Proof.** First, suppose $\lambda_*(E) = m < \infty$. Since

$$\lambda_*(E) = \sup\{\lambda(F) \mid F \subset E, F \text{ is compact}\},$$

then by definition of supremum, for all $\varepsilon_k = 1/k$, $k \in \mathbb{N}$, there is a compact set $F_k$ such that $m \geq \lambda(F_k) > m - 1/k$. Consider the set $F = \cup_{k=1}^{\infty} F_k$. Since each $F_k$ is compact (and therefore closed), then $F$ is a countable union of closed sets—i.e., $F$ is an $F_\sigma$ set. Also, $F_k \subset F \subset E$ for all $k \in \mathbb{N}$. Therefore, by monotonicity of $\lambda_*$:

$$m - \frac{1}{k} = \lambda_*(F_k) \leq \lambda_*(F) \leq \lambda_*(E) = m$$

for all $k \in \mathbb{N}$, and hence $\lambda_*(F) = \lambda_*(E)$.

Second, suppose $\lambda_*(E) = \infty$. Then for all $k \in \mathbb{N}$ there is a compact set $F_k$ such that $\lambda_*(F_k) > k$ from the supremum definition of $\lambda_*(E)$. Again, take $F = \cup F_k$ and $F$ is an $F_\sigma$ set with

$$\lambda_*(F) = \lambda_*(\cup F_k) \geq \sum \lambda_*(F_k) = \infty = \lambda_*(E)$$

where the inequality part follows from the countable superadditivity of $\lambda_*$ ∎

**Theorem 3.3.** If $F$ is a compact set, then $\lambda^*(F) = \lambda_*(F)$. In the next section, we will see that this is the definition of *measurable*. So every compact set $F$ is measurable.

**Proof.** Let $[a, b]$ be the smallest interval containing $F$. We know that $(a, b) = ((a, b) \setminus F) \cup F$ and since $\lambda^*$ is countably additive,

$$\lambda^*((a, b)) = \lambda^*(((a, b) \setminus F) \cup F) = \lambda^*((a, b) \setminus F) + \lambda^*(F)$$

or

$$\lambda^*(F) = \lambda^*((a, b)) - \lambda^*((a, b)\setminus) = b - a - \lambda^*((a, b) \setminus F) = \lambda(F) = \lambda_*(F).$$

So $F$ is measurable. ∎

**Note.** We cannot use intervals (directly) in the definition of inner measure, since set $E$ may not have any subsets which are intervals (consider $\mathbb{Q}$ or $\mathbb{R} \setminus \mathbb{Q}$). However, every set has a compact subset (since, trivially, the empty set is compact and has outer measure 0).

**Theorem 3.4.** Let $[a, b]$ be the smallest interval containing set $E$. Then

$$\lambda_*(E) = b - a - \lambda^*([a, b] \setminus E).$$

**Proof.** First, let $F \subset E$ be compact. Then $[a, b] \setminus F$ is open and $[a, b] \setminus E \subset [a, b] \setminus F$. then

$$\begin{aligned}
\lambda(F) &= b - a - \lambda([a, b] \setminus F) \text{ (definition of } \lambda \text{ for a compact set)} \\
&\leq b - a - \inf\{\lambda(G) \mid [a, b] \setminus E \subset G, G \text{ is open}\} \\
&\quad \text{(definition of infimum since } [a, b] \setminus F \\
&\quad \text{is one specific such open } G) \\
&= b - a - \lambda^*([a, b] \setminus E) \text{ (definition of } \lambda^*).
\end{aligned}$$

Since $F \subset E$ was arbitrary, taking a suprema over all such $F$ yields

$$\lambda_*(E) \leq b - a - \lambda^*([a, b] \setminus E).$$

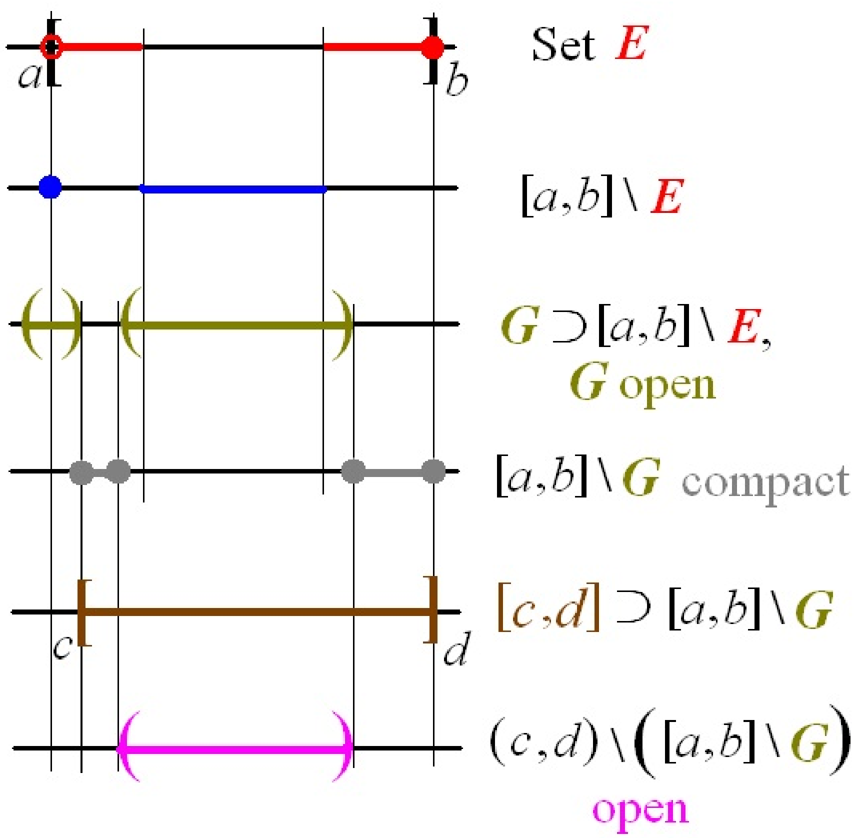
We now need to reverse this inequality.

Second, let $[a, b] \setminus E \subset G$ where $G$ is open. Then $[a, b] \setminus G$ is compact and $[a, b] \setminus G \subset E$. Then

$$\begin{aligned}
b - a - \lambda(G) &\leq b - a - \inf\{\lambda(G) \mid [a, b] \setminus E \subset G, G \text{ is open}\} \\
&\quad \text{(definition of infimum)} \\
&= b - a - \lambda^*([a, b] \setminus E) \text{ (definition of } \lambda^*),
\end{aligned}$$

or

$$\begin{aligned}
\lambda(E) &\geq \lambda([a,b] \setminus G) \text{ (definition of supremum since } [a,b] \setminus G \\
&\qquad \text{is one specific such compact set)} \\
&= (d-c) - \lambda((c,d) \setminus ([a,b] \setminus G)) \\
&\qquad \text{where } [c,d] \text{ is the smallest closed interval containing } [a,b] \setminus G) \\
&\geq (b-a) - \lambda((c,d) \setminus ([a,b] \setminus G)) \text{ (since } [c,d] \subset [a,b]).
\end{aligned}$$




If $a, b \in E$, then $(c,d) = (a,b)$ and WLOG we have $G \subset (a,b)$, so

$$(c,d) \setminus ([a,b] \setminus G) = (a,b) \setminus ([a,b] \setminus G) = (a,b) \setminus ((a,b) \setminus G) = G.$$

Then

$$\lambda_*(E) \geq b - a - \lambda((c,d) \setminus ([a,b] \setminus G)) = b - a - \lambda(G)$$

where $G$ is open and $[a,b] \setminus E \subset G$. Since $G$ was arbitrary (we have $G \subset (a,b)$ WLOG), taking the infimum over all such $G$ gives

$$\lambda_*(E) \geq b - a - \lambda^*([a,b] \setminus E).$$

Therefore when $a, b \in E$ (i.e., when $E$ contains its lub and glb),

$$\lambda_*(E) = b - a - \lambda^*([a,b] \setminus E).$$

If $a$ is not in $E$, we see that $[a,b] \setminus E$ differs from $[a,b] \setminus (E \cup \{a\})$ by only one point. Hence, from an $\varepsilon$-argument, we can show that $\lambda^*([a,b] \setminus E) = \lambda^*([a,b] \setminus (E \cup \{a\}))$ (and similarly if neither $a$ nor $b$ is in $E$) and the result follows for arbitrary $E$. ∎

**Note.** We will define a set to be *Lebesgue measurable* by always appealing to bounded portions of the set. Therefore the equation

$$\lambda_*(E) = b - a - \lambda^*([a,b] \setminus E)$$

has some implication even for unbounded sets. The important observation here is that even if we approach Lebesgue measure from an inner measure/outer measure perspective, we see that the inner measure is ultimately dependent only on the outer measure. Therefore, there is a degree of redundance in the introduction of inner measure at least as long as the above equation holds (and this is where the Carathéodory splitting condition arises in Royden and Fitzpatrick's development).

# 4. Lebesgue Measurability

**Definition** Let $E$ be a bounded subset of $\mathbb{R}$, and let $\lambda^*(E)$ and $\lambda_*(E)$ denote the outer and inner measures of $E$. If

$$\lambda_*(E) = \lambda^*(E)$$

then we say that $E$ is *Lebesgue measurable* with *Lebesgue measure* $\lambda(E) = \lambda^*(E)$. If $E$ is unbounded, we say that $E$ is *Lebesgue measurable* if $E \cap I$ is Lebesgue measurable for every finite interval $I$ and again write $\lambda(E) = \lambda^*(E)$.

**Note.** Henri Lebesgue (1875–1941) was the first to crystallize the ideas of measure and the integral studied in Part 1 of our Real Analysis 1 class. In his doctoral dissertation, *Intégrale, Longueur, Aire* ("Integral, Length, Area") of 1902, he presented the definitions of inner and outer measure equivalent to the approach of Bruckner, Bruckner, and Thomson given here. His definition of "measurable" is the same as the previous definition. Lebesgue published his results in 1902, with the same title as his dissertation, in *Annali di Matematica Pura ed Applicata*, Series 3, VII(4), 231–359. You can find this online (in French, or course) on Archive.org (accessed 4/21/2021). Carathéodory introduced his splitting condition in 1914. His approach is to outer measure and measurability in a more abstract setting. His results appeared in Über das lineare Mass von Punktmengen- eine Verallgemeinerung des Längenbegriffs ["About the linear measure of sets of points - a generalization of the concept of length"] *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse* ["News of the Society of Sciences in Göttingen , Mathematics and Physical Class"] (1914), 404–426. Carathéodory's original paper can be found online on Göttinger Digitalisierungszentrum Ein Service der SUB Göttingen (accessed 4/21/2021).

**Theorem 4.1.** $\lambda_*$ is monotone. That is, if $E_1 \subset E_2$ then $\lambda_*(E_1) \leq \lambda_*(E_2)$.

**Proof.** Let $E_a \subset E_2$. Since every compact set $F$ which is a subset of $_q$ is also a subset of $E_2$, then

$$\lambda_*(E_1) = \sup\{\lambda(F) \mid F \subset E_1, F \text{ compact}\}$$

$$\leq \sup\{\lambda(F) \mid F \subset E_2, F \text{ compact}\} = \lambda^*(E_2)$$

(since the second supremum is taken over a larger collection of real numbers than the first supremum). ∎

**Theorem 4.2.** If $\{E_k\}$ is a disjoint sequence of subsets of $\mathbb{R}$, then

$$\lambda_*\left(\bigcup_{k=1}^{\infty} E_k\right) \geq \sum_{k=1}^{\infty} \lambda_*(E_k).$$

This property is called *countable superadditivity.*

**Proof.** Let $\varepsilon > 0$. By the definition of $\lambda_*(E_k)$ in terms of a supremum, for each $k \in \mathbb{N}$ there exists a compact set $F_k \subset E_k$ such that

$$\lambda_*(E_k) - \frac{\varepsilon}{2^k} \leq \lambda_*(F_k) = \lambda(F_k),$$

a property of supremum. Next,

$$\begin{aligned} \lambda_*\left(\bigcup_{k=1}^{n} E_k\right) &\geq \lambda_*\left(\bigcup_{k=1}^{n} F_k\right) \text{ (by the monotonicity of } \lambda_*) \\ &= \lambda\left(\bigcup_{k=1}^{n} F_k\right) \text{ (since each } \cup F_k \text{ is compact and so measurable)} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{n} \lambda(F_k) \text{ (since } \lambda \text{ is countably additive)} \\
&\geq \sum_{k=1}^{n} \left( \lambda_*(E_k) + \frac{\varepsilon}{2^k} \right) \\
&= \sum_{k=1}^{n} \lambda_*(E_k) + \varepsilon \left( \sum_{k=1}^{n} \frac{1}{2^k} \right)
\end{aligned}
$$

This holds for all $n$, so

$$\lambda_* \left( \bigcup_{k=1}^{\infty} E_k \right) \geq \sum_{k=1}^{\infty} \lambda_*(E_k) + \varepsilon.$$

Next, $\varepsilon$ was arbitrary, so

$$\lambda_* \left( \bigcup_{k=1}^{\infty} E_k \right) \geq \sum_{k=1}^{\infty} \lambda_*(E_k).$$

∎

**Note.** If $E \subset \mathbb{R}$ is a bounded measurable set, and $[a, b]$ is the smallest interval containing $E$, then

$$\lambda_*(E) = (b - a) - \lambda^*([a, b] \setminus E) \text{ by Theorem 3.4}$$

or

$$\lambda^*(E) = \lambda^*([a, b]) - \lambda^*([a, b] \setminus E)$$

or

$$\lambda^*([a, b]) = \lambda^*(E) + \lambda^*([a, b] \setminus E). \quad (1)$$

Recall the Carathéodory splitting condition from Royden and Fitzpatrick:

$$\lambda^*(X) = \lambda^*(A) + \lambda^*(X \setminus A).$$

Equation (1) is simply the splitting condition applied to the set $A = [a, b]$! If $E$ is measurable and unbounded, then the condition of Lebesgue measurability implies that the splitting condition must be satisfied for all intervals. (By the additivity of $\lambda^*$, we can replace interval $[a, b]$ with any interval and say the same thing about unbounded mearsurable sets.)

**Note.** Clearly, the splitting condition implies (1) and so Royden and Fitzpatrick's approach implies the inner/outer measure approach to defining Lebesgue measure. We now need to show that the inner/outer measure approach implies Royden and Fitzpatrick's approach and the Carathéodory splitting condition. This is accomplished in the following theorem.

**Theorem 4.3.** Let $E \subset \mathbb{R}$ is a bounded measurable set (i.e., $\lambda_*(E) = \lambda^*(E)$) and let $[a, b]$ be the smallest interval containing $E$. Then for any set $A \subset \mathbb{R}$ we have

$$\lambda^*(A) = \lambda^*(A \cap E) + \lambda^*(A \setminus E).$$

**Proof.** Let $E \subset \mathbb{R}$ be a bounded measurable set and let $[a, b]$ be the smallest interval containing set $E$. Let $A$ be any subset of $[a, b]$. By Theorem 3.1, there is a $G_\delta$ set $G \supset A$ (called a measurable cover of $A$) such that $\lambda^*(G) = \lambda^*(A)$. Since $A \subset [a, b]$ and set $G$ is $G_\delta$, then WLOG we have $G \subset [a, b]$: Since $[a, b] = \bigcap_{n=1}^{\infty} \left[a + \frac{1}{n}, b - \frac{1}{n}\right]$ is $G_\delta$ and, if $G$ is not a subset of $[a, b]$, the set $G \cap [a, b]$ is a $G_\delta$ subset of $[a, b]$ and $A \subset G \cap [a, b]$. By monotonicity of $\lambda^*$, we have

$$\lambda^*(A) \leq \lambda^*(A \cap E) + \lambda^*(A \setminus E).$$

So we only need to show that

$$\lambda^*(A) \geq \lambda^*(A \cap E) + \lambda^*(A \setminus E).$$

Notice that

$$[a,b]\backslash G = [a,b]\cap G^c = ([a,b]\cup([a,b]\backslash E))\cap G^c = ([a,b]\cap G^c)\bigcup(([a,b]\backslash E)\cap G^c)$$
$$= ([a,b]\setminus G)\bigcup(([a,b]\setminus E)\setminus G)$$

and so by monotonicity of $\lambda^*$

$$\lambda^*(E\setminus G) + \lambda^*(([a,b]\setminus E)\setminus G) \geq \lambda^*([a,b]\setminus G). \ (1)$$

Since we know from Royden and Fitzpatrick that $G$ is measurable (*in the sense of Royden and Fitzpatrick*) and so $G$ satisfies the splitting condition and

$$\lambda^*(E) = \lambda^*(E\cap G) + \lambda(E\setminus G)) \ (2)$$

(the splitting condition on $G$ applied to set $E$) and

$$\begin{aligned}
\lambda^*([a,b]\setminus E) &= \lambda^*(([a,b]\setminus E)\cap([a,b]\setminus G)) + \lambda^*(([a,b]\setminus E)\setminus([a,b]\setminus G)) \\
&\quad \text{(splitting condition on } [a,b]\setminus G \text{ applied to set } [a,b]\setminus E) \\
&= \lambda^*(([a,b]\setminus G)\setminus e) + \lambda^*(G\setminus E) \text{ since } G\subset[a,b]. \ (3)
\end{aligned}$$

Since $E$ is measurable, by countable additivity

$$\lambda^*([a,b]) = \lambda^*([a,b]\cap E) + \lambda^*([a,b]\setminus E) = \lambda^*(E) + \lambda^*([a,b]\setminus E).$$

Therefore

$$\begin{aligned}
\lambda([a,b]) &= \lambda^*([a,b] - \lambda^*(E) + \lambda^*([a,b]\setminus E) \\
&= (\lambda^*(E\cap G) + \lambda^*(E\setminus G) + \lambda^*([a,b]\setminus E) \\
&\quad \text{since from } G \text{ is measurable, from (2)} \\
&= \lambda^*(E\cap G) + \lambda^*(E\setminus G) + (\lambda^*(([a,b]\setminus G)\setminus E) + \lambda^*(G\setminus E)) \\
&\quad \text{since } [a,b]\setminus \ \text{ is measurable, from (3)}
\end{aligned}$$

$$\begin{aligned}
&= (\lambda^*(E \cap G) + \lambda^*(G \setminus E)) + (\lambda^*(E \setminus G) + \lambda^*(([a,b] \setminus G) \setminus E)) \\
&\geq \lambda^*(G) + \lambda^*([a,b] \setminus G) \text{ by monotonocity, since } G = (E \cap G) \cup (G \setminus E) \\
&\quad \text{and } (([a,b] \setminus G) \setminus E) \cup (E \setminus G) = [a,b] \setminus G) \\
&= \lambda^*([a,b] \cap G)^*_\lambda([a,b] \setminus G) \text{ (since } G \subset [a,b]) \\
&= \lambda^*([a,b]) = \lambda([a,b]) \text{ since } G \text{ is measurable} \\
&\qquad \text{—splitting condition on } [a,b] \text{ applied to set} G).
\end{aligned}$$

Therefore the inequality reduces to equality and

$$\lambda^*(E \cap G) + \lambda^*(G \setminus E) + \lambda^*(E \setminus G) + \lambda^*(([a,b] \setminus G) \setminus E)$$

$$= \lambda^*(G) + \lambda^*([a,b] \setminus G).$$

Subtracting (1) from both sides yields

$$\lambda^*(E \cap G) + \lambda^*(G \setminus E) \leq \lambda^*(G). \quad (4)$$

Since $A \subset G$, we have $A \cap E \subset G \cap E$ and $A \setminus E \subset G \setminus E$, and by monotonicity

$$\begin{aligned}
\lambda^*(A \cap E) + \lambda^*(A \setminus E) &\leq \lambda^*(G \cap E) + \lambda^*(G \setminus E) \\
&\leq \lambda^*(G) \text{ (by (4))} \\
&= \lambda^*(A) \text{ (since } G \text{ is a measurable content of } A).
\end{aligned}$$

Combining this with our first inequality, we have established

$$\lambda^*(A) = \lambda^*(A \cap E) + \lambda^*(A \setminus E)$$

for all $A \subset [a,b]$. Therefore the splitting condition is satisfied on $E$ applied to arbitrary set $A \subset [a,b]$. ∎

**Note.** No where in the previous proof did we use the fact that $[a, b]$ is *the smallest* interval containing set $E$. We can therefore state:

**Corollary 1.** If $E \subset \mathbb{R}$ is a bounded measurable set (i.e., $\lambda_*(E) = \lambda^*(E)$), then for any bounded set $A$ we have

$$\lambda^*(A) = \lambda^*(A \cap E) + \lambda^*(A \setminus E).$$

**Note.** Since we (following Bruckner, Bruckner, Thomson) have defined unbounded set $E$ to be measurable if, for any finite interval $I$, set $E \cap I$ is measurable, we can extend the previous corollary by eliminating the boundedness restriction:

**Corollary 2.** If $E \subset \mathbb{R}$ is a measurable set (i.e., $\lambda_*(E) = \lambda^*(E)$), then for any set $A \subset \mathbb{R}$ we have

$$\lambda^*(A) = \lambda^*(A \cap E) + \lambda^*(A \setminus E).$$

**Note.** In conclusion, we have shown that a set $E \subset \mathbb{R}$ is measurable (i.e., $\lambda_*(E) = \lambda^*(E)$) if and only if the Carathéodory splitting condition is satisfied:

$$\lambda^*(A) = \lambda^*(A \cap E) + \lambda^*(A \setminus E).$$

Therefore the inner/outer measure definition of Lebesgue measurability (Bruckner/Bruckner/Thomson's) is equivalent to the splitting condition approach (Royden/Fitzpatrick's).

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