Some Graph, Digraph, and Mixed Graph Results Concerning Decompositions, Packings, and Coverings

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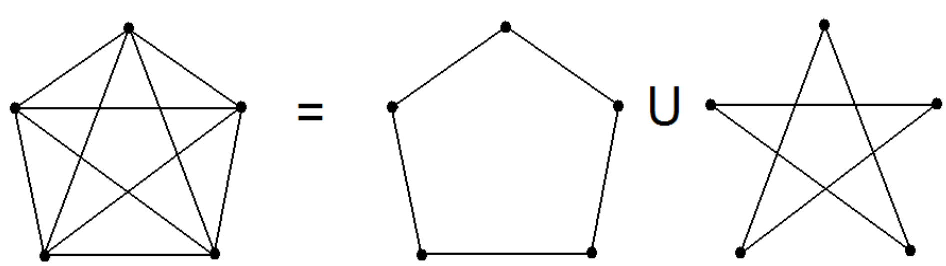
1. Decompositions

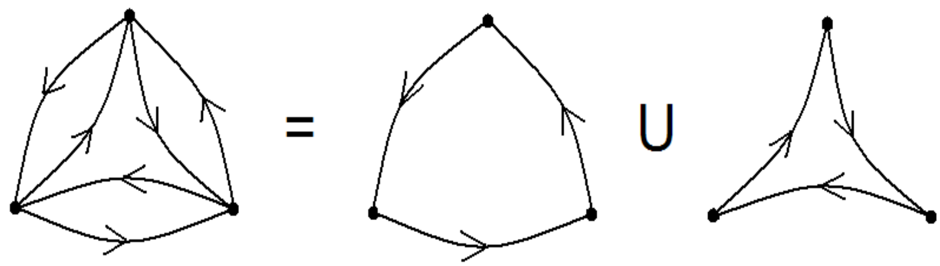
Definition. A *decomposition* of a simple graph G with isomorphic copies of graph g is a set $\{g_1, g_2, \ldots, g_n\}$ where $g_i \cong g$ and $V(g_i) \subset V(G)$ for all i, $E(g_i) \cap E(g_i) = \emptyset$ if $i \neq j$, and $\cup_{i=1}^n g_i = G$. Here, $V(G)$ is the vertex set of graph G and $E(G)$ is the edge set of graph G.

Note. Decompositions of digraphs are similarly defined (replacing edge sets with arc sets).

Example. There is a decomposition of K_5 into 5-cycles (C_5's):

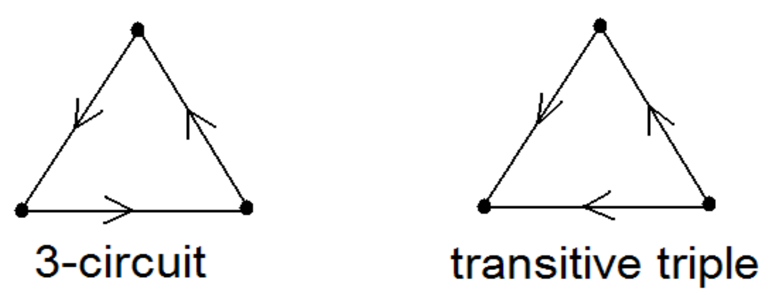


Example. There is a decomposition of the complete digraph on 3 vertices (D_3) into 3-circuits:



Note. A decomposition of K_v into C_3's is equivalent to a *Steiner triple system* of order v, denoted $STS(v)$. It is well known that a $STS(v)$ exists if and only if $v \equiv 1$ or $3 \pmod 6$.

Definition. There are two orientations of C_3, a 3-circuit and a transitive triple:



A decomposition of D_v into 3-circuits is equivalent to a *Mendelsohn triple system* of order v, $MTS(v)$. A decomposition of D_v into transitive triples is equivalent to a *directed triple system* of order v, $DTS(v)$. A $MTS(v)$ exists if and only if $v \equiv 0$ or $1 \pmod 3$, $v \neq 6$ [Mendelsohn, 1971]. A $DTS(v)$ exists if and only if $v \equiv 0$ or 1 (mod 3) [Hung and Mendelsohn, 1973].

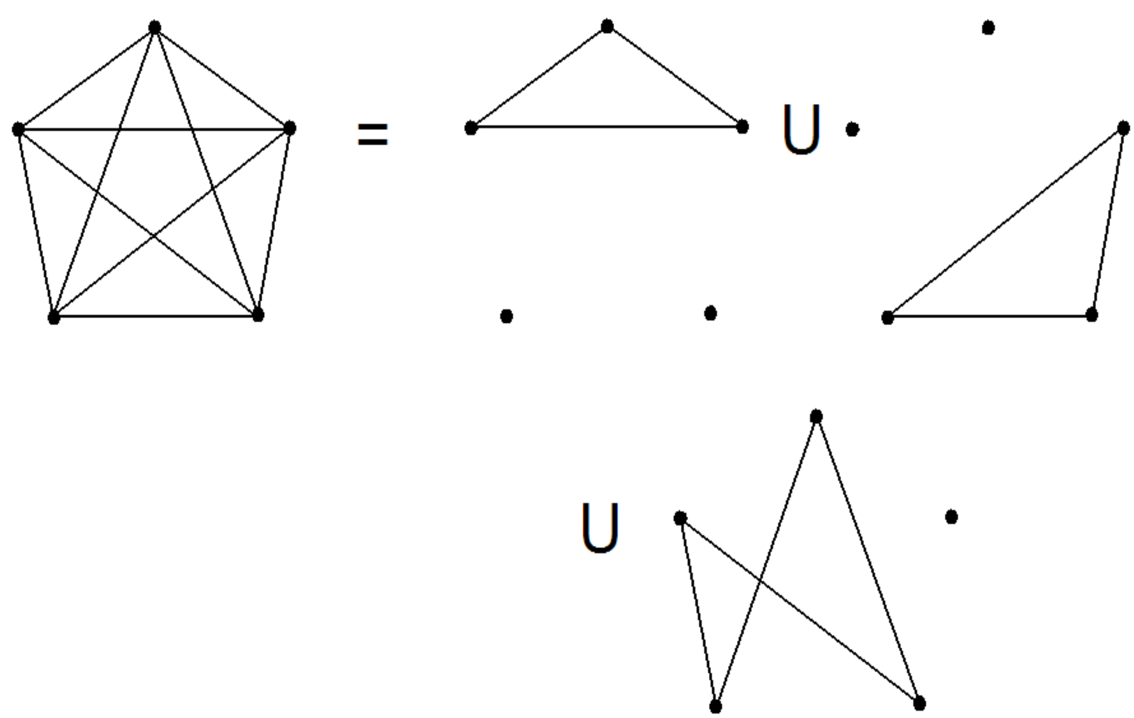
2. Packings and Coverings

Definition. A *maximal packing* of a simple graph G with isomorphic copies of graph g is a set $\{g_1, g_2, \ldots, g_n\}$ where $g_i \cong g$ and $V(g_i) \subset V(G)$ for all i, $E(g_i) \cap E(g_j) = \emptyset$ if $i \neq j$, $\cup_{i=1}^n g_i \subset G$, and

$$|E(L)| = |E(G) \setminus \cup_{i=1}^n E(g_i)|$$

is minimal. The set L is called the *leave* of the packing.

Example. A packing of K_5 with 3-cycles has a leave L with 4 edges:



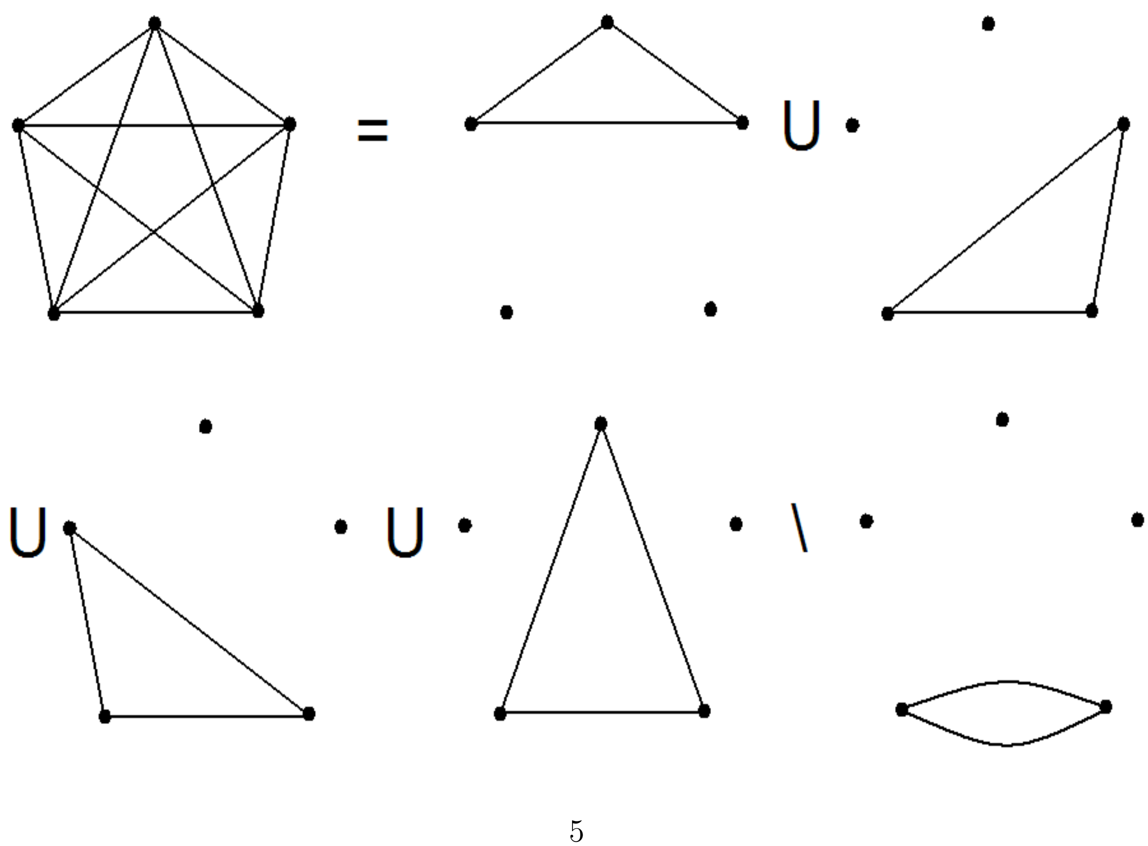
Note. Packings of the complete graph on v vertices, K_v, with graph g have been studied for g a 3-cycle [Schönheim, 1966], g a 4-cycle [Schönheim and Bialostocki, 1975], $g = K_4$ [Brouwer, 1979], and g a 6-cycle [Kennedy, 1993].

Definition. A *minimal covering* of a simple graph G with isomorphic copies of a graph g is a set $\{g_1, g_2, \ldots, g_n\}$ where $g_i \cong g$ and $V(g_i) \subset V(G)$ for all i, $G \subset \cup_{i=1}^n g_i$, and

$$|E(P)| = |\cup_{i=1}^n E(g_i) \setminus E(G)|$$

is minimal (the graph $\cup_{i=1}^n g_i$ may not be simple and $\cup_{i=1}^n E(g_i)$ may be a multiset). The graph P is called the *padding* of the covering.

Example. A covering of K_5 with 3-cycles has a padding of $2 \times K_2$:



Note. Coverings of K_v with graph g have been studied for g a 3-cycle [Fort and Hedlund, 1958], g a 4-cycle [Schönheim and Bialostocki, 1975], and g a 6-cycle [Kennedy, 1995].

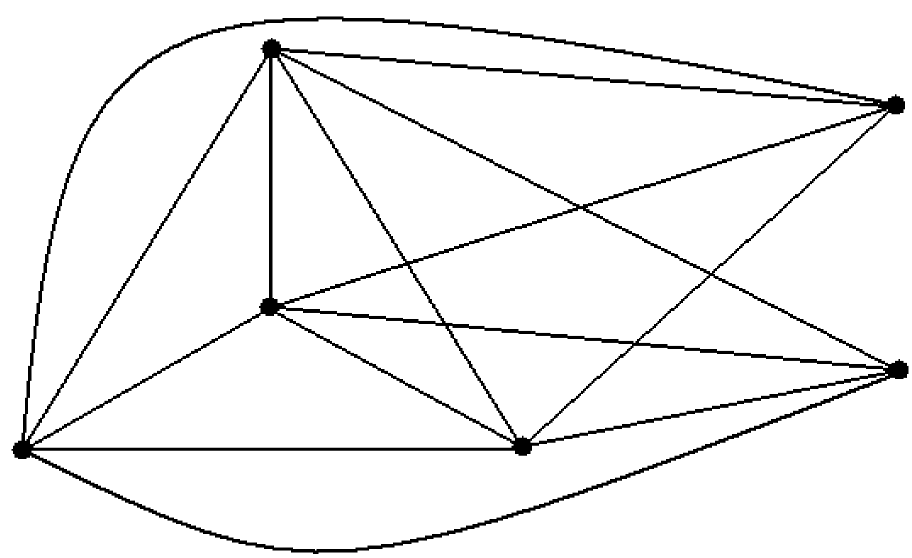
3. 4-Cycles and the Complete Graph with a Hole

Definition. The *complete graph on v vertices with a hole of size w*, denoted $K(v, w)$ is the graph with vertex set $V(K(v, w)) = V_{v-w} \cup V_w$ where $|V_{v-w}| = v - w$ and $|V_w| = w$, and with edge set

$$E(K(v, w)) = \{(a, b) \mid a \neq b, \{a, b\} \subset V_{v-w} \cup V_w \text{ and } \{a, b\} \not\subset V_w\}.$$

We let $V_{v-w} = \{1_1, 2_1, \ldots, (v - w)_1\}$ and $V_w = \{1_2, 2_2, \ldots, w_2\}$.

Example. The complete graph on 6 vertices with a hole of size 2, $K(6, 2)$, is:



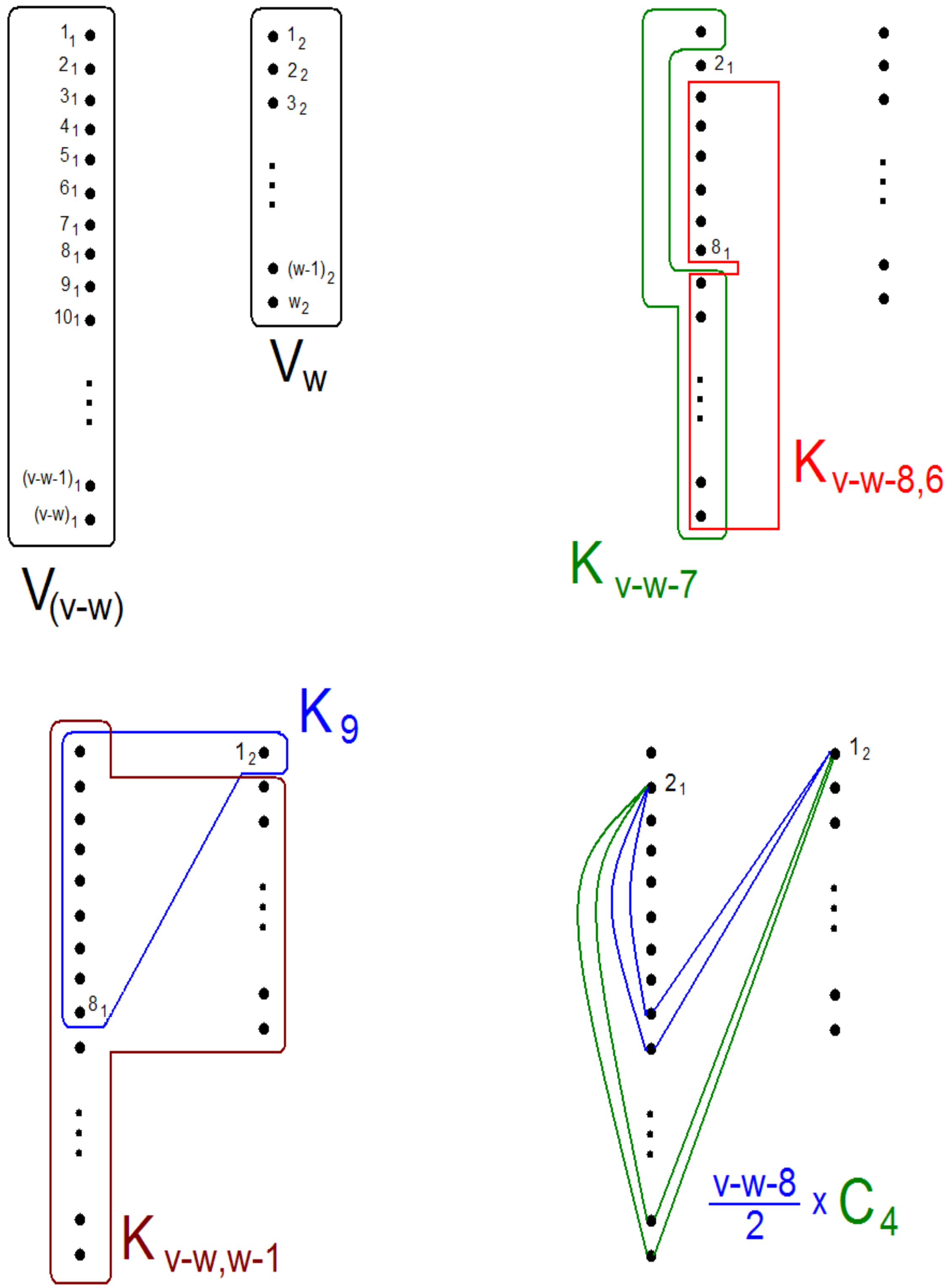
Note. It is rather well known that a 4-cycle decomposition of K_v exists if and only if $v \equiv 1 \pmod 8$ [Schönheim and Bialostocki, 1975]. It is quite easy to show that $K_{m,n}$ can be decomposed into C_4's if and only if $m \equiv n \equiv 0 \pmod 2$.

Theorem. [Gardner, Lavoie, and Nguyen, 2005] A C_4 decomposition of $K(v, w)$ exists if and only if $w \equiv 1 \pmod 2$ and $v - w \equiv 0 \pmod 8$.

Proof. Since each vertex of C_4 is even, a necessary condition is that each vertex of $K(v, w)$ must be even. A vertex of V_{v-w} is of degree $v - 1$, therefore v must be odd. A vertex of V_w is of degree $v - w$ and so $v - w$ must be even and w must be odd. The graph $K(v, w)$ has $v(v - 1)/2 - w(w - 1)/2$ edges. Since C_4 has four edges, another necessary condition for the desired decomposition is that $v(v-1)/2-w(w-1)/2 \equiv 0 \pmod 4$. Together, these conditions yield the necessary conditions of the theorem.

Now

$$K(v, w) = K_{v-w-7} \bigcup K_{v-w-8,6} \bigcup K_{v-w,w-1} \bigcup K_9 \bigcup (v-w-8)/2 \times C_4.$$



Now $v-w-7 \equiv 1 \pmod 8$ and $9 \equiv 1 \pmod 8$, so K_{v-w-7} and K_9 can be decomposed into C_4's. Next, $v-w$ and $w-1$ are even, $v-w-8$ and 6 are even, so $K_{v-w,w-1}$ and $K_{v-w-8,6}$ can be decomposed into C_4's. Therefore $K(v,w)$ can be decomposed into C_4's. ∎

Note. Schönheim and Bialostocki [1975] studied C_4 packings of K_v. Bryant and Khodkar [2000] studied C_3 packings of $K(v, w)$. We now look at C_4 packings of $K(v, w)$.

Theorem. [Gardner, Lavoie, and Nguyen, 2005] A C_4 packing of $K(v, w)$ exists if and only if:

1. if $v - w \equiv 0 \pmod 2$ and $w \equiv 1 \pmod 2$, then

$$|E(L)| = \begin{cases} 0 & \text{if} \quad v - w \equiv 0 (\text{mod } 8) \\ 3 & \text{if} \quad v - w \equiv 2 (\text{mod } 8) \\ 6 & \text{if} \quad v - w \equiv 4 (\text{mod } 8) \\ 5 & \text{if} \quad v - w \equiv 6 (\text{mod } 8), \end{cases}$$

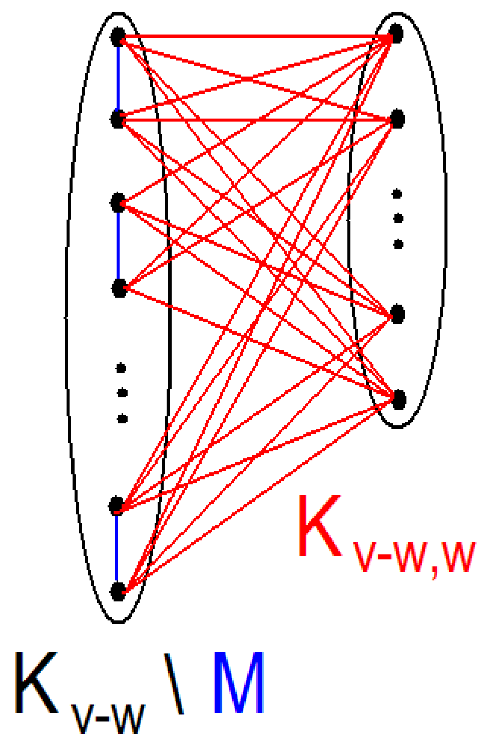
2. if $v - w \equiv 0 \pmod 2$ and $w \equiv 0 \pmod 2$, then $|E(L)| = (v - w)/2$,

3. if $v - w \equiv 1 \pmod 2$ and $w \equiv 0 \pmod 2$, then $|E(L)| = w + k$ where k is the minimum nonnegative integer such that $|E(K(v, w))| - |E(L)| \equiv 0 \pmod 4$,

4. if $v - w \equiv 1 \pmod 2$, $w \equiv 1 \pmod 2$, and $v - w \leq w$, then $|E(L)| = w + k$ where k is the minimum nonnegative integer such that $|E(K(v, w))| - |E(L)| \equiv 0 \pmod 4$, and

5. if $v - w \equiv 1$, $w \equiv 1 \pmod 2$, and $v - w > w$, then $|E(L)| = v/2 + k$ where k is the minimum nonnegative integer such that $|E(K(v, w))| - |E(L)| \equiv (\text{mod } 4)$.

"Proof." The proof consists of 17 cases. Consider the case $v - w \equiv 0 \pmod 2$ and $w \equiv 0 \pmod 2$. Each vertex of V_{v-w} is of degree $v - w - 1$ which is odd, therefore in the leave L each vertex from V_{v-w} must be of odd degree. So a packing with $|E(L)| = (v - w)/2$ would be optimal. Now $K(v, w) = (K_{v-w} \setminus M) \bigcup K_{v-w,w} \bigcup M$:



We can show (using difference methods) that $K_{v-w} \setminus M$ can be decomposed into C_4's. Since $v - w$ and w are both even, then $K_{v-w,w}$ can be decomposed into C_4's. We then have an optimal packing with leave $L = M$ where M is a matching on K_{v-w} and so $|E(L)| = (v - w)/2$. ∎

Theorem. [Gardner, Lavoie, and Nguyen, 2005] A C_4 covering of $K(v,w)$ exists if and only if:

1. if $v-w \equiv 0 \pmod 2$, $v-w>2$, and $w \equiv 1 \pmod 2$, then

$$|E(P)| = \begin{cases} 0 & \text{if} \quad v-w \equiv 0 (\text{mod } 8) \\ 5 & \text{if} \quad v-w \equiv 2 (\text{mod } 8) \\ 2 & \text{if} \quad v-w \equiv 4 (\text{mod } 8) \\ 3 & \text{if} \quad v-w \equiv 6 (\text{mod } 8), \end{cases}$$

2. if $v-w \equiv 0 \pmod 4$ and $w \equiv 0 \pmod 2$, then $|E(P)| = (v-w)/2$,

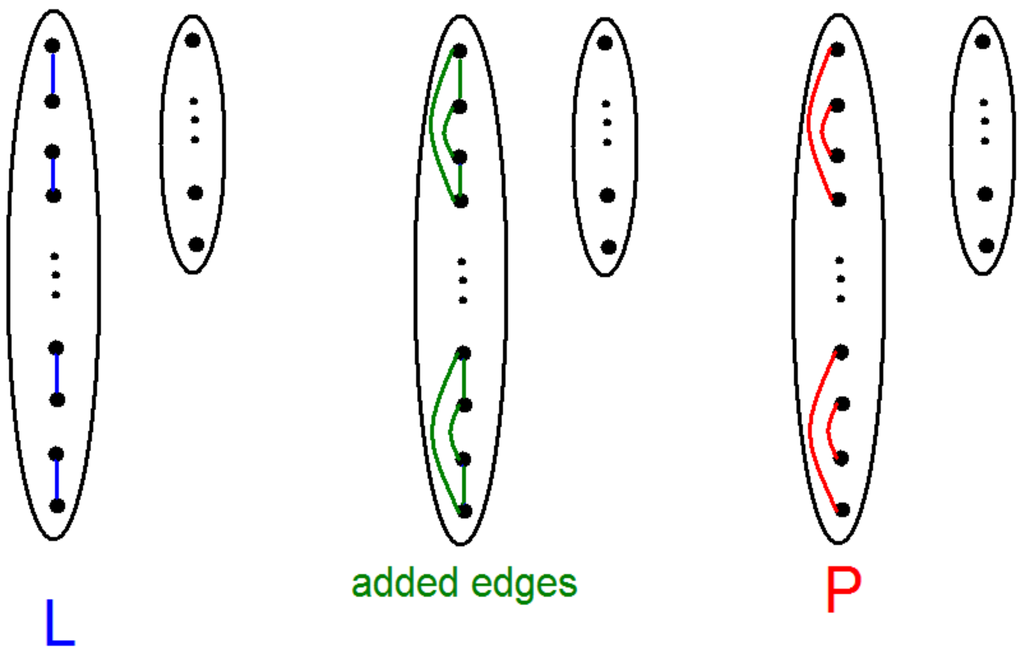
3. if $v-w \equiv 2 \pmod 4$ and $w \equiv 0 \pmod 2$, then $|E(P)| = (v-w)/2+2$,

4. if $v-w \equiv 1 \pmod 2$, $v-w>1$, and $w \equiv 0 \pmod 2$, then $|E(P)| = w+k$ where k is the minimum nonnegative integer such that $|E(K(v,w)| + |E(P)| \equiv 0 \pmod 4$,

5. if $v-w \equiv 1 \pmod 2$, $v-w>1$, $w \equiv 1 \pmod 2$, and $v-w \leq w$, then $|E(P)| = w+k$ where k is the minimum nonnegative integer such that $|E(K(v,w))| + |E(P)| \equiv 0 \pmod 4$, and

6. if $v-w \equiv 1$, $w \equiv 1 \pmod 2$, and $v-w>w$, then $|E(P)| = v/2+k$ where k is the minimum nonnegative integer such that $|E(K(v,w))| + |E(P)| \equiv \pmod 4$.

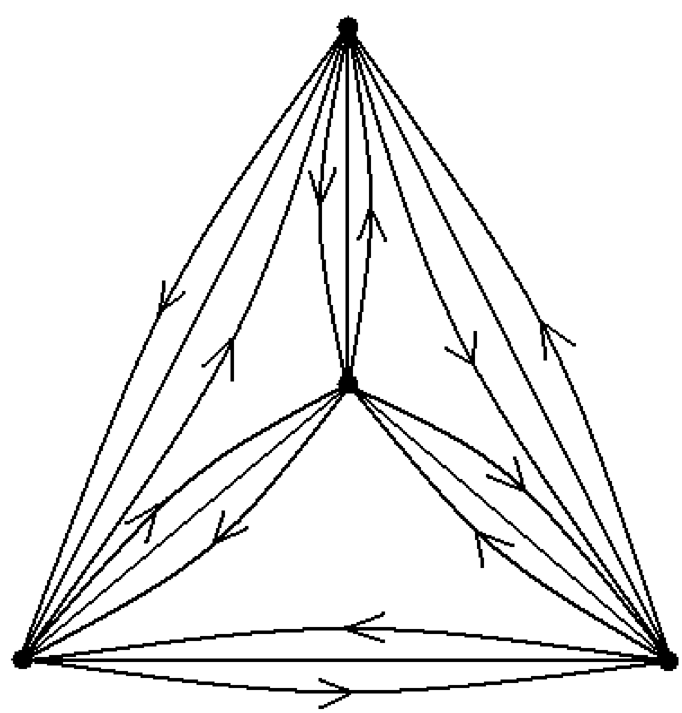
“Proof.” The proof consists of 22 cases. Consider the case $v-w \equiv 0 \pmod 2$ and $w \equiv 0 \pmod 2$. Each vertex of V_{v-w} is of degree $v-w-1$ which is odd, therefore in the padding P each vertex from V_{v-w} must be of odd degree. So a covering with $|E(P)| = (v-w)/2$ would be optimal. We take the packing described above with the leave L a matching on V_{v-w}. We then add C_4's as below, and have a padding P which is also a matching on V_{v-w} and hence $|E(P)| = (v-w)/2$. ∎



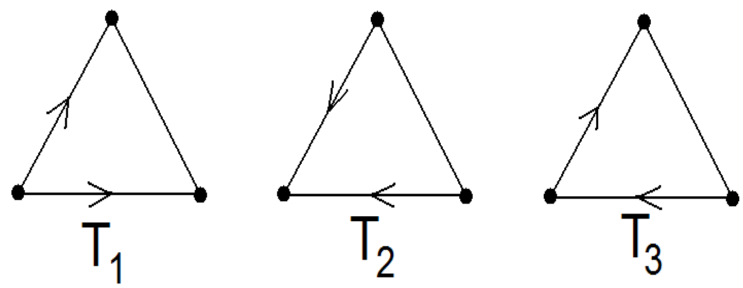
4. Some Results Concerning Mixed Graphs

Definition. A *mixed graph* on v vertices is an ordered pair (V, C) where V is a set of vertices, $|V| = v$, and C is a set of ordered and unordered pairs, denoted $[x, y]$ and (x, y) respectively, of elements of V. An ordered pair $[x, y] \in C$ is called an *arc* of (V, C) and an unordered pair $(x, y) \in C$ is called an *edge* of graph (V, C). The *complete mixed graph* on v vertices, denoted M_v, is the mixed graph (V, C) where, for every pair of distinct vertices $v_1, v_2 \in V$, we have $\{[v_1, v_2], [v_2, v_1], (v_1, v_2)\} \subset C$.

Example. The mixed graph M_4 is:



Note. Since M_v has twice as many arcs as edges, we are inspired to study triple system based on complete mixed graphs and the following:



Definition. A decomposition of M_v into T_i's is a T_i *triple system* or order v.

Theorem. [Gardner, 1999] A T_i triple system of order v exists for all $i \in \{1, 2, 3\}$ and $v \equiv 1 \pmod 2$, except for $i = 3$ and $v \in \{3, 5\}$.

Note. A study of packing and covering M_v with T_i ($i \in \{1, 2, 3\}$) is currently underway by Bobga and Gardner.

Definition. Let G be a graph and $\gamma = \{g_1, g_2, \ldots, g_n\}$ be a g decomposition of G. An *automorphism* of this decomposition is a permutation of $V(G)$ which fixes set γ. An automorphism of digraph and mixed graph decompositions are similarly defined.

Definition. Consider a permutation on a set of size v. The permutation is said to be *cyclic* if it consists of a single cycle of length v. It is *bicyclic* if it consists of two disjoint cycles of lengths N_1 and N_2 where $v = N_1 + N_2$.

Theorems. A cyclic $STS(v)$ exists if and only if $v \equiv 1$ or 3 (mod 6), $v \neq 9$ [Peltesohn, 1939]. A bicyclic $STS(v)$ where $v = N_1 + N_2$ admitting an automorphism whose disjoint cyclic decomposition is a cycle of length N_1, where $N_1 > 1$, and a cycle of length N_2 exists if and only if $N_1 \equiv 1$ or 3 (mod 6), $N_1 \neq 9$, $N_1 \mid N_2$, and $v = N_1 + N_2 \equiv 1$ or 3 (mod 6) [Calahan-Zijlstra and Gardner, 1994].

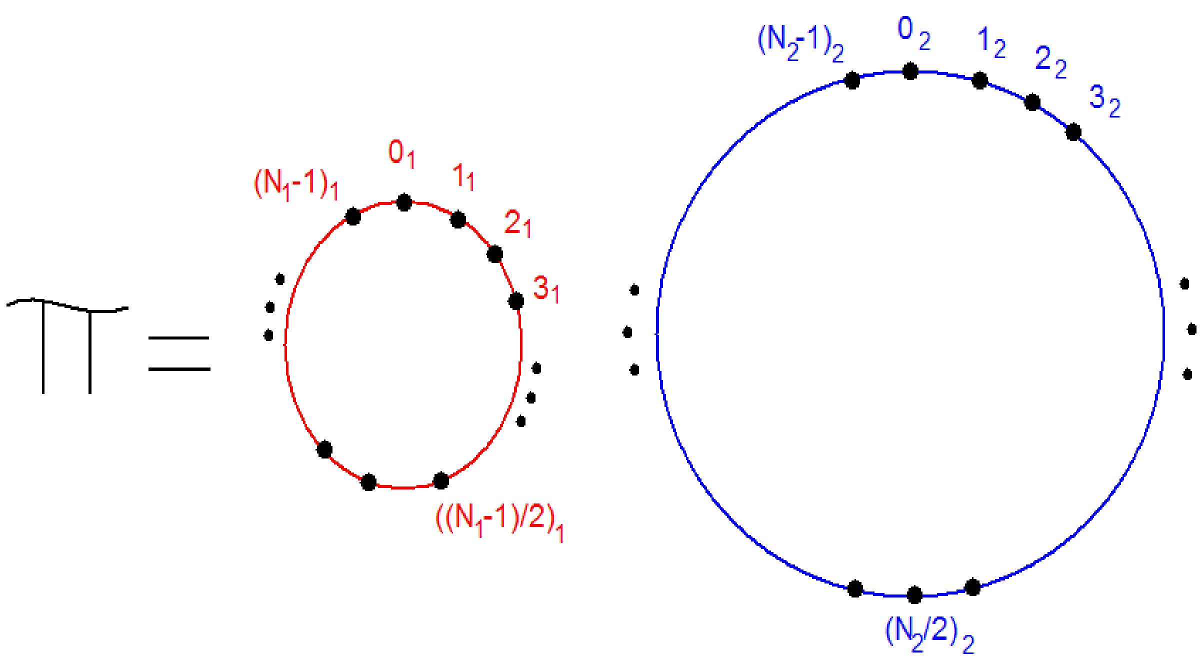
Theorems. A cyclic $DTS(v)$ exists if and only if $v \equiv 1$, 4, or 7 (mod 12) [Colbourn and Colbourn, 1982]. A bicyclic $DTS(v)$ admitting an automorphism consisting of two cycles each of length $v/2$ exists if and only if $v \equiv 4$ (mod 6). A bicyclic $DTS(v)$ admitting an automorphism consisting of a cycle of length N_1 and a cycle of length N_2, where $v = N_1 + N_2$, exists if and only if $N_1 \equiv 1$, 4, or 7 (mod 12) and $N_2 = kN_1$ where $k \equiv 2$ (mod 3) [Gardner, 1998].

Theorem. A cyclic $MTS(v)$ exists if and only if $v \equiv 1$ or 3 (mod 6), $v \neq 9$ [Colbourn and Colbourn, 1981]. To the best of my knowledge, bicyclic MTS's have not been studied.

Theorem. [Gardner, 1999] A cyclic T_i triple system of order v exists for all $i \in \{1, 2, 3\}$ and $v \equiv 1$ (mod 2), except for $i = 3$ and $v \in \{3, 5\}$.

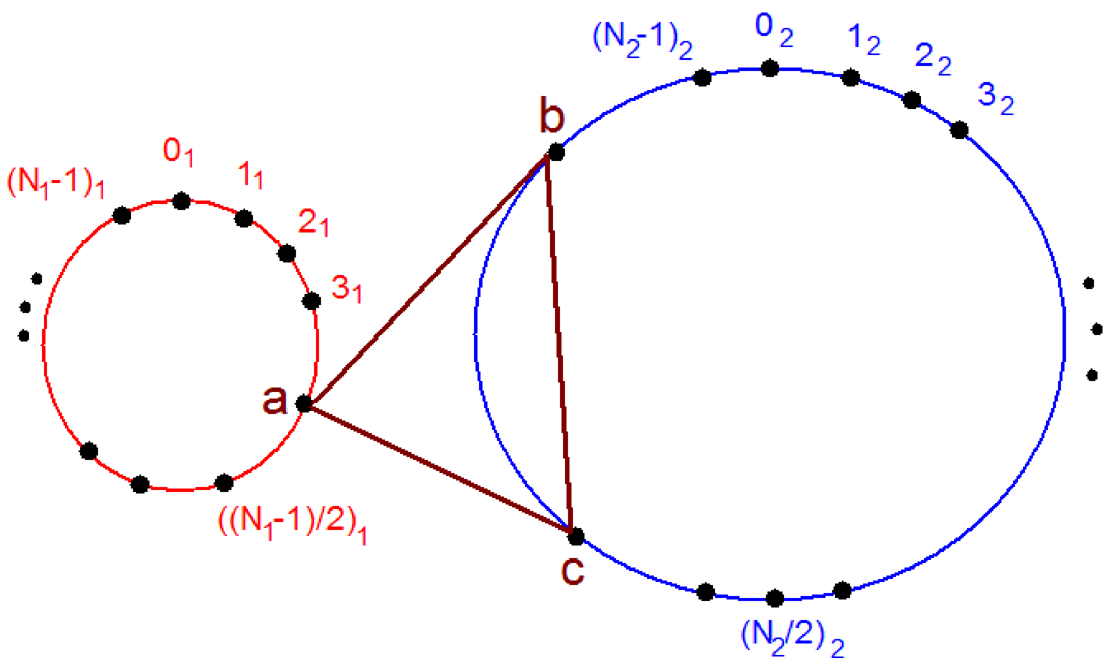
Theorem. [Bobga and Gardner, 2005] A bicyclic T_i triple system, where $i \in \{1, 2\}$, exists admitting an automorphism consisting of a cycle of length N_1 and a cycle of length N_2, where $N_1 < N_2$, if and only if $N_1 \equiv 1 \pmod 2$, $N_1 \mid N_2$, and $v = N_1 + N_2 \equiv 1 \pmod 2$. A bicyclic T_3 triple system does not exist.

"Proof." Let π be a bicyclic automorphism of a T_i system where π consists of disjoint cycles of lengths N_1 and N_2:



Notice that π^{N_1} fixes the points $\{0_1, 1_1, \ldots, (N_1 - 1)_1\}$. Therefore these points form a cyclic subsystem of order N_1 and hence $N_1 \equiv 1 \pmod 2$.

Now consider some T_i with vertex set $\{a, b, c\}$ where $a \in \{0_1, 1_1, \ldots, (N_1-1)_1\}$ and $\{b, c\} \subset \{0_2, 1_2, \ldots, (N_2-1)_2\}$:



When we apply π^{N_2} to this triple, we see that $\pi^{N_2}(b) = b$ and $\pi^{N_2}(c) = c$, and hence $\pi^{N_2}(a) = a$. That is, N_2 is a multiple of N_1. This established the necessary conditions. Sufficiency is established through difference methods. ∎

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