# Spin waves

The Heisenberg model of magnetism supports *magnon excitations*, or *spin waves*, which may be identified by mapping the three spin components $\hat{S}_{i,x}$, $\hat{S}_{i,y}$, and $\hat{S}_{i,z}$ on the boson creation and annihilation operators $\hat{a}_i^\dagger$ and $\hat{a}_i$, respectively. The transition from spin to bosonic operators, obeying the commutator relation $[\hat{a}_i, \hat{a}_j] = \delta_{ij}$ is accomplished by use of the *Holstein-Primakoff transformation* [1]. In what follows, we will introduce and motivate this approach, and further show that it leads to a *magnon representation* of the Heisenberg Hamiltonian [2]. In more detail:

We establish a relation between spin operators and bosonic operators $\hat{a}$ and $\hat{a}^\dagger$ by defining

$$\hat{S}_{i,z} \equiv S_i \hat{1} - \hat{a}_i^\dagger \hat{a}_i \tag{1}$$

Using this equation, one may describe the $z$ component of the spin at some site $i$ in terms of the number of its maximum, $S_{i,z} = S_i$ as well as the number $n_i$ of spins flipped to reduce this maximum to the actual eigenvalue of the spin $z$ component, $S_{i,z}$:

$$\hat{S}_{i,z} |S_{i,z}\rangle = (S_i - n_i) |S_{i,z}\rangle. \tag{2}$$

The number $n_i$ is eigenvalue of a boson occupation operator, according to

$$\hat{a}_i^\dagger \hat{a}_i |n_i\rangle = n_i |n_i\rangle, \tag{3}$$

such that $n_i$ is to be associated with the number of pairs formed by two spins with opposite orientation. Eq. (2) implies:

$$S_{i,z} = S_i - n_i. \tag{4}$$

Therefore, we have for the spin ladder operator $\hat{S}_i^+$

$$\begin{aligned} \hat{S}_i^+ |n_i\rangle &= (S_i(S_i+1) - (S_i - n_i)(S_i - n_i + 1))^{\frac{1}{2}} |n_i - 1\rangle \\ &= (2S_i n_i - n_i^2 + n_i)^{\frac{1}{2}} |n_i - 1\rangle \\ &= \sqrt{2S_i}(1 - \frac{n_i - 1}{2S})^{\frac{1}{2}} \sqrt{n_i} |n_i - 1\rangle \end{aligned} \tag{5}$$

The logic of the bosonic states generated by $\hat{a}_i^\dagger$ dictates that the action of $\hat{S}_i^+$, breaking up one spin-up/spin-down pair, reduces the boson number by one. In the following step, we make use of Eq. (3) and of the identity

$$\hat{a}_i |n_i\rangle = \sqrt{n_i} |n_i - 1\rangle \tag{6}$$

to express $\hat{S}_i^+$ through the bosonic creation and annihilation operators and arrive at

$$\hat{S}_i^+ = \sqrt{2S - \hat{a}_i^\dagger \hat{a}_i} \hat{a}_i. \tag{7}$$

A parallel relation holds for the step-down operator $\hat{S}_i^-$, namely

$$\hat{S}_i^- = \hat{a}_i^\dagger \sqrt{2S - \hat{a}_i^\dagger \hat{a}_i}. \tag{8}$$

Note that the transformation law (1) is readily obtained from $\hat{S}_{i,z}^2 = S(S+1) - \hat{S}_{i,x}^2 - \hat{S}_{i,y}^2$ in conjunction with

$$\hat{S}_{i,x}^2 + \hat{S}_{i,y}^2 = \frac{1}{2}(\hat{S}_i^+ \hat{S}_i^- + \hat{S}_i^- \hat{S}_i^+), \tag{9}$$

as well as the commutation relation of the bosonic operators and Eqs.(78).

In the following we discuss two basic applications of the Holstein-Primakoff formalism to the Heisenberg Hamiltonian. Specifically, we derive magnon dispersion relations for one-dimensional ferromagnets and antiferromagnets.

## 0.1 The spectrum of a ferromagnetic chain

We write the Heisenberg Hamiltonian for a one-dimensional arrangement of evenly spaced, ferromagnetically ordered spins in terms of the spin components $\hat{S}_{i,x}$, $\hat{S}_{i,y}$, and $\hat{S}_{i,z}$:

$$\hat{H}_H = -J \sum_n \hat{\mathbf{S}}_n \hat{\mathbf{S}}_{n+1} = -J \sum_n [\hat{S}_{n,z} \hat{S}_{n+1,z} + \frac{1}{2}(\hat{S}_n^+ \hat{S}_{n+1}^- + \hat{S}_n^- \hat{S}_{n+1}^+)] \tag{10}$$

Substituting for $\hat{S}_{i,z}$, $\hat{S}_i^+$, and $\hat{S}_i^-$ by use of Eqs. (1), (7), and (8), one finds

$$\hat{H}_H = 2SJ \sum_n \hat{a}_n^\dagger \hat{a}_n - JS \sum_n (\hat{a}_{n+1}^\dagger \hat{a}_n + \hat{a}_n^\dagger \hat{a}_{n+1}) \tag{11}$$

The Fourier transforms of the lattice operators $\hat{a}_i^\dagger$ and $\hat{a}_i$ engender *magnon creation* and *magnon annihilation* operators, respectively [2]:

$$\begin{aligned} \hat{c}_k^\dagger &= \frac{1}{\sqrt{N}} \sum_n \hat{a}_n^\dagger e^{indk}, \\ \hat{c}_k &= \frac{1}{\sqrt{N}} \sum_j \hat{a}_n e^{-indk}, \end{aligned} \tag{12}$$

and, correspondingly:

$$\begin{aligned} \hat{a}_n^\dagger &= \frac{1}{\sqrt{N}} \sum_k \hat{c}_k^\dagger e^{-indk}, \\ \hat{a}_n &= \frac{1}{\sqrt{N}} \sum_k \hat{c}_k e^{indk}. \end{aligned} \tag{13}$$

Here, $N$ stands for the overall number of lattice sites, and $d$ for the spacing between adjacent sites. One verifies easily that the magnon operators satisfy the bosonic commutator relation, $[\hat{c}_k, \hat{c}_{k'}] = \delta_{k,k'}$. Inserting Eqs. (13) into Eq. (11) yields

$$\hat{H}_H = 2SJ \sum_k (1 - \cos(kd)) \hat{c}_k^\dagger \hat{c}_k. \tag{14}$$

Cast into this form, the Heisenberg Hamiltonian for a ferromagnetic system is determined by a number operator $\hat{N}_k = \hat{c}_k^\dagger \hat{c}_k$ that counts magnons. Thus, we may write the Hamiltonian in a still more compact form, namely:

$$\hat{H}_H = \sum_k \hbar\omega_k \hat{N}_k, \tag{15}$$

with the dispersion relation

$$E(k) = \hbar\omega_k = 2SJ(1 - \cos(kd)). \tag{16}$$

We emphasize that the limit $kd << 1$:

$$E(k) \approx SJ(kd)^2 \tag{17}$$

yields, to a good approximation, a parabolic function $E(k)$. This is the case of a free particle.

## 0.2 The spectrum of an antiferromagnetic chain

The antiferromagnetic chain is governed by the same Heisenberg Hamiltonian as the ferromagnetic chain, (10), except for a switch of sign:

$$\hat{H}_H = J \sum_n \hat{\mathbf{S}}_n \hat{\mathbf{S}}_{n+1} = J \sum_n [\hat{S}_{n,z} \hat{S}_{n+1,z} + \frac{1}{2}(\hat{S}_n^+ \hat{S}_{n+1}^- + \hat{S}_n^- \hat{S}_{n+1}^+)]. \tag{18}$$

While the treatment of the antiferromagnetic chain parallels that of the ferromagnetic counterpart it differs from the latter as fluctuations around the maximum $\hat{S}_{i,z}$ quantum number, $+S$, alternate with those around the minimum quantum number, $-S$. To accommodate the defining magnetic order of the antiferromagnetic chain, one describes it as a combination of two interpenetrating lattices, A and B. The Holstein-Primakoff transformation as given by Eqs. (1), (7), and (8) is then extended to encompass two groups of boson operators, one acting on A sites, and the other [2, 3] on B sites:

$$\begin{aligned} \hat{S}_{m,z}^A &= S_m \hat{1} - \hat{a}_m^\dagger \hat{a}_m, \\ \hat{S}_m^{A,+} &= \sqrt{2S - \hat{a}_m^\dagger \hat{a}_m}\, \hat{a}_m, \\ \hat{S}_m^{A,-} &= \hat{a}_m^\dagger \sqrt{2S - \hat{a}_m^\dagger \hat{a}_m}, \end{aligned} \tag{19}$$

and

$$
\begin{aligned}
\hat{S}^B_{j,z} &= -S_j \hat{1} + \hat{b}^\dagger_j \hat{b}_j, \\
\hat{S}^{B,+}_j &= \hat{b}^\dagger_j \sqrt{2S - \hat{b}^\dagger_j \hat{b}_j} \hat{b}_j, \\
\hat{S}^{B,-}_j &= \sqrt{2S - \hat{b}^\dagger_j \hat{b}_j} \hat{b}_j.
\end{aligned} \tag{20}
$$

The two sets of indices, $\{m\}$ and $\{j\}$, are reserved for sites of type A and B, respectively. Inserting the spin operators $(19, 20)$ into the Heisenberg Hamiltonian (18) yields an expression of great complexity. For the limiting case of small fluctuations, however, one obtains a Hamiltonian analogous to the ferromagnetic counterpart. This limit is defined by the condition $\langle \hat{a}^\dagger_m \hat{a}_m \rangle, \langle \hat{b}^\dagger_j \hat{b}_j \rangle << 2S$, i.e. the boson number at the two sites is small as compared with twice the spin quantum number $S$. This results in

$$
\begin{aligned}
\hat{S}^A_{m,z} &= S_m \hat{1} - \hat{a}^\dagger_m \hat{a}_m, \\
\hat{S}^{A,+}_m &= \sqrt{2S} \hat{a}_m, \\
\hat{S}^{A,-}_m &= \hat{a}^\dagger_m \sqrt{2S},
\end{aligned} \tag{21}
$$

and correspondingly for the B sublattice. This linearization leads to the following the Heisenberg Hamiltonian:

$$
\begin{aligned}
\hat{H}_H = J \sum_m [\frac{2S}{2} (\hat{a}_m \hat{b}_{m+1} + \hat{a}^\dagger_m \hat{b}^\dagger_{m+1}) + S(\hat{a}^\dagger_m \hat{a}_m + \hat{b}^\dagger_{m+1} \hat{b}_{m+1}) - S^2] \\
+ J \sum_j [\frac{2S}{2} (\hat{a}_j \hat{b}_{j+1} + \hat{a}^\dagger_j \hat{b}^\dagger_{j+1}) + S(\hat{a}^\dagger_j \hat{a}_j + \hat{b}^\dagger_{j+1} \hat{b}_{j+1}) - S^2].
\end{aligned} \tag{22}
$$

Any component higher than quadratic in the bosonic creation and annihilation operators has been omitted in this formula. Introducing the Fourier transforms of the we write in analogy to Eqs. (12) and (13):

$$
\begin{aligned}
\hat{a}_k &= \frac{1}{\sqrt{N_A}} \sum_m e^{-imdk} \hat{a}_m, \\
\hat{b}_k &= \frac{1}{\sqrt{N_B}} \sum_j e^{-ijdk} \hat{a}_j,
\end{aligned} \tag{23}
$$

with $N_A + N_B = N$. Correspondingly:

$$
\begin{aligned}
\hat{a}_i &= \frac{1}{\sqrt{N_A}} \sum_k e^{imdk} \hat{a}_k, \\
\hat{b}_j &= \frac{1}{\sqrt{N_B}} \sum_k e^{ijdk} \hat{b}_k,
\end{aligned} \tag{24}
$$

and equivalently for the creation operators. It should be noted that the number of orthogonal modes distinguished by the index $k$ is identical with the number of lattice sites, $N_A = N/2$ for the A lattice, $N_B = N/2$ for the B lattice, and $N$ in total. We express $\hat{H}_H$ in terms of the magnon operators $\hat{a}_k$ and $\hat{b}_k$ and simplify, using the orthogonality relation $\sum_m e^{i(k-k')md} = N_A \delta_{k,k'} = \frac{N}{2}\delta_{k,k'}$. In order to apply this relation, we carry out the $k$ summation for the operators $\hat{b}/\hat{b}^\dagger$, in some cases, over the index $-k'$ rather than $k'$. This imports the operators $\hat{b}_{-k}$ and $\hat{b}^\dagger_{-k}$. It follows:

$$\hat{H}_H = -NJS^2 + 2JS\sum_k [\cos(kd)(\hat{a}_k\hat{b}_{-k} + \hat{a}^\dagger_k\hat{b}^\dagger_{-k}) + \hat{a}^\dagger_k\hat{a}_k + \hat{a}^\dagger_k\hat{b}_k] \tag{25}$$

The last two operator products are in diagonal form, but not the first two. While, for the ferromagnetic case, $H_H$ turned out to be diagonalized by Fourier transformation, this is different for the antiferromagnetic alternative. We address this situation by applying a *Bogoliubov transformation* on the operator pair $\hat{a}, \hat{b}$:

$$\begin{aligned}\hat{a}_k &= v_k\hat{\alpha}_k + w_k\hat{\beta}^\dagger_{-k},\\ \hat{b}_k &= v_k\hat{\beta}_k + w_k\hat{\alpha}^\dagger_{-k},\end{aligned} \tag{26}$$

where

$$\begin{aligned}\hat{\alpha}_k &= v_k\hat{a}_k - w_k\hat{b}^\dagger_{-k},\\ \hat{\beta}_k &= v_k\hat{b}_k - w_k\hat{a}^\dagger_{-k}.\end{aligned} \tag{27}$$

The new operators $\hat{\alpha}, \hat{\beta}$ are constrained by the bosonic commutator relations

$$\begin{aligned}[\hat{\alpha}_k, \alpha^\dagger_{k'}] = [\hat{\beta}_k, \beta^\dagger_{k'}] &= \delta_{k,k'},\\ [\hat{\alpha}_k, \beta_{k'}] &= 0.\end{aligned} \tag{28}$$

These conditions imply:

$$v_{-k} = v_k, w_{-k} = w_k, v_k^2 - w_k^2 = 1. \tag{29}$$

Recasting $H_H$ by use of relations (26) and (28), we arrive at

$$\begin{aligned}\hat{H}_H = -NJS^2 + JS\sum_k [(2\cos(kd)v_kw_k + v_k^2 + w_k^2)(\hat{\alpha}^\dagger_k\hat{\alpha}_k + \hat{\beta}^\dagger_k\hat{\beta}_k)\\ +2(\cos(kd)v_kw_k + w_k^2) + (\cos(kd)\{v_k^2 + w_k^2\} + 2v_kw_k)(\hat{\alpha}_k\hat{\beta}_{-k} + \hat{\alpha}^\dagger_k\hat{\beta}^\dagger_{-k})].\end{aligned} \tag{30}$$

The condition

$$\cos(kd)(v_k^2 + w_k^2) + 2v_k w_k = 0 \tag{31}$$

is satisfied by $v_k = \cosh\theta_k, w_k = \sinh\theta_k$, and $\tanh 2\theta_k = -\cos(kd)$. It obviously diagonalizes the Hamiltonian since it annihilates the non-diagonal terms. This leaves

$$\begin{aligned}\hat{H}_H = -NJS^2 - NJS + \sum_k [\hbar\omega_k(\hat{\alpha}_k^\dagger\hat{\alpha}_k + \hat{\beta}_k^\dagger\hat{\beta}_k + 1) \\ = E_0 + \sum_k \hbar\omega_k(\hat{\alpha}_k^\dagger\hat{\alpha}_k + \hat{\beta}_k^\dagger\hat{\beta}_k)\end{aligned} \tag{32}$$

with

$$\hbar\omega_k = JS\sqrt{1-\cos^2(kd)} = JS\sin(kd) \tag{33}$$

and

$$E_0 = -NJS^2 - NJS + \sum_k \hbar\omega_k \tag{34}$$

From Eq.(33), it follows that for $k \to 0$, the antiferromagnetic chain is characterized by linear dispersion:

$$\omega_k \propto |k|, \mathrm{as}\, k \to 0, \tag{35}$$

in contrast to its ferromagnetic analogue, where quadratic dispersion was found (see Eq. (17)).

Spin waves may be probed in the laboratory by a variety of experimental techniques, including inelastic neutron or photon scattering, such as Brillouin, Raman, and X-ray scattering, further electron scattering, such as spin resolved electron energy loss spectroscopy (SREELS), or ferromagnetic resonance (FMR).

# Bibliography

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