Coordinates and Change of Basis

Linear Algebra

MATH 2010

- **Definition:** If $B = \{v_1, v_2, ..., v_n\}$ is a basis for a vector space V and $x = c_1v_1 + c_2v_2 + ... + c_nv_n$, then c_1, c_2, ..., c_n are called the *coordinates of x relative to the basis B*. The coordinate vector is denoted

$$[x]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

- **Example:** The vector

$$x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

is the coordinate vector with respect to the standard basis $\{[1,0],[0,1]\}$

$$x = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- **Example:** Let $B = \{[1,0],[1,2]\}$. Assume

$$[x]_B = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Find x relative to the standard basis $B' = \{[1,0],[0,1]\}$.

$$[x]_{B'} = 3\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

- **Example:** Given

$$x = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

relative to the standard basis, find $[x]_{B'}$ where $B' = \{[1,0,1],[0,-1,2],[2,3,-5]\}$.

We need to find c_1, c_2, and c_3 such that

$$x = c_1u_1 + c_2u_2 + c_3u_3$$

or

$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} + c_3\begin{bmatrix} 2 \\ 3 \\ -5 \end{bmatrix}$$

This leads to the linear system

$$\begin{array}{rcrcrcr} 1 & = & c_1 & & & + & 2c_3 \\ 2 & = & & - & c_2 & + & 3c_3 \\ -1 & = & c_1 & + & 2c_2 & - & 5c_3 \end{array}$$

Solving the system we have

$$\left[\begin{array}{rrr|r} 1 & 0 & 2 & 1 \\ 0 & -1 & 3 & 2 \\ 1 & 2 & -5 & -1 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & 2 & 1 \\ 0 & 1 & -3 & -2 \\ 0 & 2 & -7 & -2 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & 2 & 1 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & -1 & 2 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

So,

$$[x]_{B'} = \begin{bmatrix} 5 \\ -8 \\ -2 \end{bmatrix}$$

- **Transition Matrix** We can write

$$[x]_{B'} = P^{-1}[x]_B$$

where P is a transition matrix from B' to B or P^{-1} is a transition matrix from B to B'.

- **Theorem:** Let $B = \{v_1, v_2, ..., v_n\}$ and $B' = \{u_1, u_2, ..., u_n\}$ be two basis for $\Re^n$. Then the transition matrix P^{-1} from B to B' can be found by using Gauss-Jordan elimination on the matrix

$$[B'|B] \rightarrow [I_n|P^{-1}]$$

- **Example:** Let

$$B = \{[1,0],[1,2]\}$$

and

$$B' = \{[1,0],[0,1]\}$$

Then the transition matrix is found by reducing:

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \end{array}\right]$$

Notice, the left hand side is already I_2, so

$$P^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

Notice that if B' is the standard basis, then $P^{-1} = B$!

For example, if

$$[x]_B = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Then

$$[x]_{B'} = P^{-1}[x]_B = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

- **Example:** Let

$$B = \{[1,0,0],[0,1,0],[0,0,1]\}$$

and

$$B' = \{[1,0,1],[0,-1,2],[2,3,-5]\}$$

Find the transition matrix from B to B' and use it to find $[x]_{B'}$ where

$$[x]_B = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

To find the matrix P^{-1} from B to B':

$$[B'|B] = \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & 3 & 0 & 1 & 0 \\ 1 & 2 & -5 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & -1 & 0 \\ 0 & 2 & -7 & -1 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 & 2 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 4 & 2 \\ 0 & 1 & 0 & 3 & -7 & -3 \\ 0 & 0 & 1 & 1 & -2 & -1 \end{array}\right]$$

So,

$$[x]_{B'} = P^{-1}[x]_B = \begin{bmatrix} -1 & 4 & 2 \\ 3 & -7 & -3 \\ 1 & -2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ -8 \\ -2 \end{bmatrix}$$

Notice that if B is the standard basis, then $P^{-1} = (B')^{-1}$!

- **Example:** Find the transition matrix from B to B' for the following bases for $\Re^2$:

$$B = \{[-3, 2], [4, -2]\}$$

and

$$B' = \{[-1, 2], [2, -2]\}$$

Ans:

$$P^{-1} = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}$$