Determinants
Linear Algebra
MATH 2010

- Determinants can be used for a lot of different applications. We will look at a few of these later. First of all, however, let's talk about how to compute a determinant.
- **Notation:** The determinant of a *square* matrix A is denoted by $|A|$ or $\det(A)$.
- **Determinant of a 2x2 matrix**: Let A be a 2x2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

  Then the determinant of A is given by

$$|A| = ad - bc$$

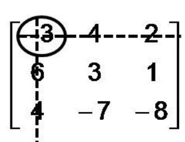
  Recall, that this is the same formula used in calculating the inverse of A:

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

  if and only if $ad - bc = |A| \neq 0$. We will later see that if the determinant of any square matrix $A \neq 0$, then A is invertible or nonsingular.
- **Terminology:** For larger matrices, we need to use cofactor expansion to find the determinant of A. First of all, let's define a few terms:
  - **Minor:** A *minor*, M_{ij}, of the element a_{ij} is the determinant of the matrix obtained by deleting the i^{th} row and j^{th} column.
    * **Example:** Let

$$A = \begin{bmatrix} -3 & 4 & 2 \\ 6 & 3 & 1 \\ 4 & -7 & -8 \end{bmatrix}$$

      Then to find M_{11}, look at element $a_{11} = -3$. Delete the entire column and row that corresponds to $a_{11} = -3$, see the image below.



      Then M_{11} is the determinant of the remaining matrix, i.e.,

$$M_{11} = \begin{vmatrix} 3 & 1 \\ -7 & -8 \end{vmatrix} = -8(3) - (-7)(1) = -17.$$

* **Example:** Similarly, to find M_{22} can be found looking at the element a_{22} and deleting the same row and column where this element is found, i.e., deleting the second row, second column:

$$\begin{bmatrix} -3 & 4 & 2 \\ 6 & 3 & 1 \\ 4 & -7 & -8 \end{bmatrix}$$

  Then

$$M_{22} = \begin{vmatrix} -3 & 2 \\ 4 & -8 \end{vmatrix} = -8(-3) - 4(-2) = 16.$$

- **Cofactor:** The cofactor, C_{ij} is given by

$$C_{ij} = (-1)^{i+j} M_{ij}.$$

  Basically, the cofactor is either M_{ij} or $-M_{ij}$ where the sign depends on the location of the element in the matrix. For that reason, it is easier to know the pattern of cofactors instead of actually remembering the formula. If you start in the position corresponding to a_{11} with a positive sign, the sign of the cofactor has an alternating pattern. You can see this by looking at a matrix containing the sign of the cofactors:

$$\begin{bmatrix} + & - & + & - & + & \cdots & \cdots \\ - & + & - & + & - & \cdots & \cdots \\ + & - & + & - & + & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

- **Determinant Using Cofactor Expansion:**
  - **Formula:** The determinant of a square nxn matrix $A$is found by
    * Expanding along the i^{th} row using the formula:

$$|A| = \sum_{j=1}^{n} a_{ij} C_{ij} = a_{i1} C_{i1} + a_{i2} C_{i2} + \cdots + a_{in} C_{in}$$

    * Expanding along the j^{th} column using the formula:

$$|A| = \sum_{i=1}^{n} a_{ij} C_{ij} = a_{1j} C_{1j} + a_{2j} C_{2j} + \cdots + a_{nj} C_{nj}$$

  - **Example:** Let

$$A = \begin{bmatrix} -3 & 4 & 2 \\ 6 & 3 & 1 \\ 4 & -7 & -8 \end{bmatrix}$$

    Then if we expand along the 1st row, we have

$$\begin{aligned} |A| &= -3 \begin{vmatrix} 3 & 1 \\ -7 & -8 \end{vmatrix} - (4) \begin{vmatrix} 6 & 1 \\ 4 & -8 \end{vmatrix} + 2 \begin{vmatrix} 6 & 3 \\ 4 & -7 \end{vmatrix} \\ &= -3[-8(3) - (-7)(1)] - 4[-8(6) - 4(1)] + 2[-7(6) - 4(3)] \\ &= -3(-17) - 4(-52) + 2(-54) \\ &= 151 \end{aligned}$$

- **Example:** Let

$$A = \begin{bmatrix} 1 & 3 & 1 & 4 \\ 0 & 5 & 0 & 5 \\ -2 & 4 & 1 & 2 \\ 0 & 6 & -1 & 0 \end{bmatrix}$$

Then

$$\begin{aligned} |A| &= 1\begin{vmatrix} 5 & 0 & 5 \\ 4 & 1 & 2 \\ 6 & -1 & 0 \end{vmatrix} - 0\begin{vmatrix} 3 & 1 & 4 \\ 4 & 1 & 2 \\ 6 & -1 & 0 \end{vmatrix} + (-2)\begin{vmatrix} 3 & 1 & 4 \\ 5 & 0 & 5 \\ 6 & -1 & 0 \end{vmatrix} - 0\begin{vmatrix} 3 & 1 & 4 \\ 5 & 0 & 5 \\ 4 & 1 & 2 \end{vmatrix} \\ &= 1\left[5\begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} - 0\begin{vmatrix} 4 & 2 \\ 6 & 0 \end{vmatrix} + 5\begin{vmatrix} 4 & 1 \\ 6 & -1 \end{vmatrix}\right] - 0\,[\ldots] - 2\left[-5\begin{vmatrix} 1 & 4 \\ -1 & 0 \end{vmatrix} + 0\begin{vmatrix} 3 & 4 \\ 6 & 0 \end{vmatrix} - 5\begin{vmatrix} 3 & 1 \\ 6 & -1 \end{vmatrix}\right] - 0 \\ &= 1[5(2) - 0 + 5(-10)] - 2[-5(4) + 0 - 5(-9)] \\ &= 1(-40) - 2(25) \\ &= -90 \end{aligned}$$

- **Example:** Try to find the determinant of

$$A = \begin{bmatrix} 5 & 3 & 0 & 6 \\ 4 & 6 & 4 & 12 \\ 0 & 2 & -3 & 4 \\ 0 & 1 & -2 & 2 \end{bmatrix}$$

Ans: 0.

- **Properties of Determinants:** Let A and B be nxn matrices, then
  - $|AB| = |A||B|$
  - $|cA| = c^n|A|$
  - $|A| = |A^T|$
  - If A^{-1} exists, $|A^{-1}| = \frac{1}{|A|}$
  - $|A + B| \neq |A| + |B|$

- **Theorem:**
  A square matrix A is invertible if and only if $|A| \neq 0$.

- **Theorem:**
  If A is a square matrix and *any* one of the following conditions are true, then $|A| = 0$.

  1. An entire row (or column) consists of zeros.
  2. Two rows (or columns) are equal.
  3. One row (or column) is a multiple of another row (or column).

  - **Example:** If

$$A = \begin{bmatrix} 4 & 3 & -2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & -7 & 13 & 12 \\ 6 & -2 & 5 & 6 & 7 \\ 1 & 4 & 2 & 0 & 9 \end{bmatrix}$$

then $|A| = 0$ because it has a row of zeros.

- **Example:** If

$$A = \begin{bmatrix} 2 & 3 & -5 \\ 1 & -2 & 4 \\ -3 & 6 & -12 \end{bmatrix}$$

then $|A| = 0$ because the third row is a multiple of the second row.

- **Equivalent Statements:** If A is an nxn matrix, then the following are equivalent:

  (a) A is invertible
  (b) $Ax = 0$ has only the trivial solution
  (c) The reduced row-echelon form of A is I_n
  (d) A is expressible as a product of elementary matrices.
  (e) $Ax = b$ is consistent for every nx1 matrix b
  (f) $Ax = b$ has exactly one solution for every nx1 matrix b
  (g) $|A| \neq 0$.

- **Example:** Does the following system have a unique solution?

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & - & x_3 & = & x_4 \\ 2x_1 & - & x_2 & + & x_3 & = & 6 \\ 3x_1 & - & 2x_2 & + & 2x_3 & = & 0 \end{array}$$

There is no need to solve this system to answer this question. Given the equivalent statements above, the system has a unique solution if and only if $|A| \neq 0$. Here

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \\ 3 & -2 & 2 \end{bmatrix}$$

Notice that the second and third column are multiples of one another. Therefore, $|A| = 0$, so there is no unique solution.

- **Cramer's Rule:**

  - **Introduction of Cramer's rule:** Cramer's rule can be used to solve for just one variable (without solving for all the variable) in a linear system $Ax = b$ for which there is a unique solution, i.e., when $|A| \neq 0$. Let's look at a simple 2 variable case and then we can generalize this for n-variable systems. Consider the system

$$\begin{array}{rcrcl} a_{11}x_1 & + & a_{12}x_2 & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & = & b_2 \end{array}$$

Using techniques from college algebra, we can solve for x_2 by multiply each equation by an appropriate term and then add to get cancel out x_1:

$$\begin{array}{rcrcl} -a_{21}(a_{11}x_1 & + & a_{12}x_2 & = & b_1) \\ a_{11}(a_{21}x_1 & + & a_{22}x_2 & = & b_2) \end{array}$$

Then adding, we get:

$$(a_{11}a_{22} - a_{21}a_{12})x_2 = a_{11}b_2 - a_{21}b_1$$

or

$$x_2 = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{21}a_{12}} = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} = \frac{|A_2|}{|A|}$$

where A_2 is the matrix formed by replacing the *second* column of A by b, i.e.,

$$A_2 = \begin{bmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{bmatrix}$$

Similarly,

$$x_1 = \frac{|A_1|}{|A|}$$

where

$$A_1 = \begin{bmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{bmatrix}$$

– **Example:** Use Cramer's rule to solve the system

$$\begin{array}{rcrcr} 2x_1 & - & x_2 & = & -10 \\ 3x_1 & + & 2x_2 & = & -1 \end{array}$$

Here

$$A = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} \text{ and } b = \begin{bmatrix} -10 \\ -1 \end{bmatrix}$$

We see from above that $|A| = 2(2) - (3)(-1) = 7$. Also

$$A_1 = \begin{bmatrix} -10 & -1 \\ -1 & 2 \end{bmatrix}$$

with $|A_1| = -10(2) - (-1)(-1) = -21$ and

$$A_2 = \begin{bmatrix} 2 & -10 \\ 3 & -1 \end{bmatrix}$$

with $|A_2| = (2)(-1) - (-10)(3) = 28$. Thus,

$$x_1 = \frac{|A_1|}{|A|} = \frac{-21}{7} = -3$$

and

$$x_2 = \frac{|A_2|}{|A|} = \frac{28}{7} = 4$$

– **Generalization:** If a system of n linear equations in n variables has a coefficient matrix with a nonzero determinant $|A|$, then the solution to the system is given by

$$x_1 = \frac{|A_1|}{|A|}, \quad x_2 = \frac{|A_2|}{|A|}, \quad \cdots \quad x_n = \frac{|A_n|}{|A|}$$

– **Example:** Consider the system

$$\begin{array}{rcrcrcr} 4x_1 & - & 2x_2 & + & 3x_3 & = & -2 \\ 2x_1 & + & 2x_2 & + & 5x_3 & = & 16 \\ 8x_1 & - & 5x_2 & - & 2x_3 & = & 4 \end{array}$$

Find x_2 without finding x_1 or x_3.
Using Cramer's rule, we know

$$x_2 = \frac{|A_2|}{|A|}$$

where

$$A = \begin{bmatrix} 4 & -2 & 3 \\ 2 & 2 & 5 \\ 8 & -5 & -2 \end{bmatrix}$$

and

$$A_2 = \begin{bmatrix} 4 & -2 & 3 \\ 2 & 16 & 5 \\ 8 & 4 & -2 \end{bmatrix}$$

Using the either the permutation method or cofactor expansion, we can find $|A| = -82$ and $|A_2| = -656$. Thus

$$x_2 = \frac{|A_2|}{|A|} = \frac{-656}{-82} = 8.$$

- **Adjoints:**
  - **Cofactors revisited:** Recall C_{ij} denotes the cofactor associated with the element a_{ij} of the matrix A. $C_{ij} = (-1)^{i+j} M_{ij}$ where M_{ij} is the determinant of the matrix obtained by deleting the i^{th} row and j^{th} column of A.
  - **Matrix of cofactors of** A is given by

$$\begin{bmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{bmatrix}$$

  - **Adjoint of** A**:** The adjoint of A is the transpose of the matrix of cofactors (given above) and is denoted by $adj(A)$:

$$adj(A) = \begin{bmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{bmatrix}^T$$

  - **Finding** A^{-1} **using Adjoints:** If A is invertible, then

$$A^{-1} = \frac{1}{|A|} adj(A)$$

    * **Example:** Find A^{-1} using $adj(A)$ where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 2 & 2 & 2 \end{bmatrix}$$

First, find $|A|$. Using method of cofactors along the first column, you get:

$$|A| = 1 \begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix} - 0 \begin{vmatrix} 2 & 3 \\ 2 & 2 \end{vmatrix} + 2 \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} = 4 - 0 - 10 = -6$$

Next, find the matrix of cofactors:

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} +\begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix} & -\begin{vmatrix} 0 & -1 \\ 2 & 2 \end{vmatrix} & +\begin{vmatrix} 0 & 1 \\ 2 & 2 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 2 & 2 \end{vmatrix} & +\begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} \\ +\begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 0 & -1 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 4 & -2 & -2 \\ 2 & -4 & 2 \\ -5 & 1 & 1 \end{bmatrix}$$

Find $adj(A)$ by taking the transpose of the matrix of cofactors above:

$$adj(A) = \begin{bmatrix} 4 & -2 & -2 \\ 2 & -4 & 2 \\ -5 & 1 & 1 \end{bmatrix}^T = \begin{bmatrix} 4 & 2 & -5 \\ -2 & -4 & 1 \\ -2 & 2 & 1 \end{bmatrix}$$

Find $A^{-1} = \frac{1}{|A|} adj(A)$:

$$A^{-1} = \frac{1}{-6} \begin{bmatrix} 4 & 2 & -5 \\ -2 & -4 & 1 \\ -2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} & -\frac{1}{3} & \frac{5}{6} \\ \frac{1}{3} & \frac{2}{3} & -\frac{1}{6} \\ \frac{1}{3} & -\frac{1}{3} & -\frac{1}{6} \end{bmatrix}$$