Diagonalization
Linear Algebra
MATH 2010

- **The Diagonalization Problem:** For a nxn matrix A, the diagonalization problem can be stated as, does there exist an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix?
- **Terminology:** If such a P exists, then A is called **diagonalizable** and P is said to **diagonalize** A.
- **Theorem** If A is a nxn matrix, then the following are equivalent:
  1. A is diagonalizable.
  2. A has n linearly independent eigenvectors.
- **Procedure for Diagonalizing a Matrix:**

Step 1 Find n linearly independent eigenvectors of A, say $\mathbf{p}_1$, $\mathbf{p}_2$, ..., $\mathbf{p}_n$.

Step 2 Form the matrix P having $\mathbf{p}_1$, $\mathbf{p}_2$, ..., $\mathbf{p}_n$ as its column vectors.

Step 3 The matrix $P^{-1}AP$ will then be diagonal with λ_1, λ_2, ..., λ_n as its diagonal entries, where λ_i is the eigenvalue corresponding to $\mathbf{p}_i$, for $i = 1, 2, ..., n$.

- **Example:** Find a matrix P, if possible, that diagonalizes

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

The eigenvalues and eigenvectors are given by $\lambda = 1$ with corresponding eigenvector

$$\mathbf{p}_1 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

and $\lambda = 2$ with corresponding eigenvectors

$$\mathbf{p}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \text{ and } \mathbf{p}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Since the matrix is 3x3 and has 3 eigenvectors, then A is diagonalizable and

$$P = \begin{bmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

and

$$P^{-1}AP = \begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

- **Example:** From the section on eigenvalues, we determined that $\lambda_1 = 1$ and $\lambda_2 = 2$ are eigenvalues of

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix}$$

with corresponding eigenvectors

$$p_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } p_2 = \begin{bmatrix} -7 \\ 4 \\ 1 \end{bmatrix}$$

Since there are only 2 basis vectors for the eigenspace of A, and A is a 3x3 matrix, A is **not** diagonalizable.

- **Theorem:** If $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_k$ are eigenvectors of A corresponding to distinct eigenvalues of λ_1, λ_2, ..., λ_k, then $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\}$ is a linearly independent set.
- **Theorem:** If an nxn matrix A has n distinct eigenvalues, then A is diagonalizable.
- **Example:** Find P, if possible that diagonalizes

  $$A = \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}$$

  and find the corresponding diagonal matrix D. We have eigenvalues $\lambda_1 = 3$ with corresponding eigenvector

  $$\mathbf{p}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

  and $\lambda_2 = -1$ with corresponding eigenvector

  $$\mathbf{p}_2 = \begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$$

  Therefore,

  $$P = \begin{bmatrix} 1 & -1/3 \\ 1 & 1 \end{bmatrix}$$

  and

  $$D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$$

- **Example:** Find P, if possible that diagonalizes

  $$A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix}$$

  and find the corresponding diagonal matrix D. The three eigenvalues for A are $\lambda_1 = 3$, $\lambda_2 = 2$, and $\lambda_3 = -2$. The eigenvector for $\lambda_1 = 3$:

  $$\mathbf{p}_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

  The eigenvector for $\lambda_2 = 2$:

  $$\mathbf{p}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

  The eigenvector for $\lambda_3 = -2$:

  $$\mathbf{p}_3 = \begin{bmatrix} 1/4 \\ -1/4 \\ 1 \end{bmatrix}$$

  So,

  $$P = \begin{bmatrix} -1 & -1 & \frac{1}{4} \\ 1 & 0 & -\frac{1}{4} \\ 1 & 1 & 1 \end{bmatrix}$$

  and

  $$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

- **Definition:** If λ_0 is an eigenvalue of an nxn matrix A, then the dimension of the eigenspace (the number of eigenvectors) corresponding to λ_0 is called the **geometric multiplicity** of λ_0. The number of times that $\lambda - \lambda_0$ appears as a factor in the characteristic polynomial of A is called the **algebraic multiplicity**.

- **Theorem:** If A is a square matrix, then
  1. For every eigenvalue of A, the geometric multiplicity is less than or equal to the algebraic multiplicity
  2. A is diagonalizable if and only if, for every eigenvalue, the geometric multiplicity is equal to the algebraic multiplicity.

- **Computing powers:** If A is an nxn matrix and P is an invertible matrix, then

$$(P^{-1}AP)^2 = P^{-1}APP^{-1}AP = P^{-1}A^2P$$

  More generally,

$$(P^{-1}AP)^k = P^{-1}APP^{-1}AP = P^{-1}A^kP$$

  Therefore, if A is diagonalizable, then

$$P^{-1}A^kP = (P^{-1}AP)^k = D^k$$

  thus

$$A^k = PD^kP^{-1}$$

- **Example:** Find A^{13} for

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

  We found previously that

$$P = \begin{bmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

  and

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

  Therefore,

$$A^{13} = PD^{13}P^{-1} = \begin{bmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1^{13} & 0 & 0 \\ 0 & 2^{13} & 0 \\ 0 & 0 & 2^{13} \end{bmatrix} \begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -8190 & 0 & -16382 \\ 8191 & 8192 & 8191 \\ 8191 & 0 & 16383 \end{bmatrix}$$

- **Definition** An nxn matrix A is *similar* to an nxn matrix Q if there exists an invertible matrix P such that $P^{-1}AP = Q$.