Eigenvalues
Linear Algebra
MATH 2010

- **Importance:** Eigenvalues is an important concept from Linear Algebra. Eigenvalues are used in population growth, differential equations, engineering, science and statistics to name a few places.

- **The Eigenvalue Problem:** For a nxn matrix A, the eigenvalue problem can be stated as, does there exist a nonzero vector x and a scalar λ such that

$$Ax = \lambda x$$

  In other words, Ax is a scalar multiple of x.

- **Terminology:** λ is called an **eigenvalue** of A and x is the corresponding **eigenvector**.

- **Example:** Verify that $\lambda = 5$ is an eigenvalue of

$$A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$$

  with corresponding eigenvector

$$x = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

  To do this, we need to check if $Ax = \lambda x$.

$$Ax = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 15 \\ 5 \end{bmatrix}$$

  and

$$\lambda x = 5 \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 15 \\ 5 \end{bmatrix}$$

  These are equal, so $\lambda = 5$ is an eigenvalue of A with corresponding eigenvector x.

- **Example:** Verify that $\lambda_1 = 1$ and $\lambda_2 = 2$ are eigenvalues of

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix}$$

  with corresponding eigenvectors

$$x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } x_2 = \begin{bmatrix} -7 \\ 4 \\ 1 \end{bmatrix}$$

  To verify λ_1:

$$Ax_1 = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

  and

$$\lambda_1 x_1 = 1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

  So, λ_1 is an eigenvalue of A with corresponding eigenvector x_1.

  To verify λ_2:

$$Ax_1 = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} -7 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -14 \\ 8 \\ 2 \end{bmatrix}$$

and

$$\lambda_2 x_2 = 2\begin{bmatrix} -7 \\ 4 \\ 1 \end{bmatrix}\begin{bmatrix} -14 \\ 8 \\ 2 \end{bmatrix}$$

So, λ_2 is an eigenvalue of A with corresponding eigenvector x_2.

- **Finding eigenvalues:** The eigenvalue problem asks if there exists a nonzero vector x and scalar λ such that

$$Ax = \lambda x$$

If we arrange the equation above, we have

$$\lambda x - Ax = (\lambda I - A)x = 0$$

So, we want to know if there exists a scalar λ and nonzero vector x which satisfies this homogeneous system. Recall that a homogeneous system $Bx = 0$ always has a solution. It has a unique solution, the trivial solution $x = 0$, if $|B| \neq 0$. In this problem, we are looking for nozero vectors x, so we are looking for a solution to the system $Bx = 0$ where $|B| = 0$. This is the only way to get a nontrivial solution. If

$$|\lambda I - A| = 0$$

then $\lambda I - A$ is singular and there exists a nontrivial solution to $(\lambda I - A)x = 0$. Therefore, we need to solve

$$|\lambda I - A| = 0$$

for λ. This is called the **characteristic equation**. All values λ which satisfy the characteristic equation are called **eigenvalues** of A.

- **Example:** Let

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}$$

Find all the eigenvalues of A. First of all, then find the matrix $\lambda I - A$.

$$\lambda I - A = \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} \lambda - 2 & -1 \\ -3 & \lambda \end{bmatrix}$$

Then

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 \\ -3 & \lambda \end{vmatrix} = \lambda(\lambda - 2) - 3$$

Then the characteristic equation is

$$\lambda(\lambda - 2) - 3 = 0$$

Solving this equation for λ:

$$\begin{aligned} \lambda(\lambda - 2) - 3 &= 0 \\ \lambda^2 - 2\lambda - 3 &= 0 \\ (\lambda - 3)(\lambda + 1) &= 0 \end{aligned}$$

Solving, we get $\lambda_1 = 3$ and $\lambda_2 = -1$

- **Finding eigenvectors:** Once you have the eigenvalues for A, then to find the corresponding eigenvectors, you solve the system

$$(\lambda I - A)x = 0$$

for x. Remember, since there will not be a unique solution, there will be ininitely many solutions (since we have found λ such that $\lambda I - A$ is singular). Therefore, there should ALWAYS be a paramter. If not, you have done something wrong - go back and check that you found the correct λ.

- **Back to example:** We found $\lambda_1 = 3$ and $\lambda_2 = -1$ are eigenvalues for

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}$$

Let's first find the corresponding eigenvector for $\lambda_1 = 3$: We need to first find $\lambda I - A = 3I - A$ which is given by

$$3I - A = \begin{bmatrix} 3-2 & -1 \\ -3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -3 & 3 \end{bmatrix}$$

Now solve $(3I - A)x = 0$:

$$\left[\begin{array}{rr|r} 1 & -1 & 0 \\ -3 & 3 & 0 \end{array}\right] \to_{R_2 \leftrightarrow R_2 + 3R_1} \left[\begin{array}{rr|r} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Thus, $x_2 = t$ is a free variable and $x_1 = x_2 = t$. Therefore,

$$x_1 = \begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Therefore, eigenvectors corresponding to $\lambda_1 = 3$ are scalar multiples of

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For $\lambda_2 = -1$, we need to first find $\lambda I - A = -I - A$ which is given by

$$-I - A = \begin{bmatrix} -1-2 & -1 \\ -3 & -1 \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ -3 & -1 \end{bmatrix}$$

Now solve $(-I - A)x = 0$:

$$\left[\begin{array}{rr|r} -3 & -1 & 0 \\ -3 & -1 & 0 \end{array}\right] \to_{R_1 \leftrightarrow \frac{-1}{3} R_1} \left[\begin{array}{rr|r} 1 & 1/3 & 0 \\ -3 & -1 & 0 \end{array}\right] \to_{R_2 \leftrightarrow R_2 + 3R_1} \left[\begin{array}{rr|r} 1 & 1/3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Thus, $x_2 = t$ is a free variable and $x_1 = -1/3x_2 = -1/3t$. Therefore,

$$x_2 = \begin{bmatrix} -1/3t \\ t \end{bmatrix} = t \begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$$

Therefore, eigenvectors corresponding to $\lambda_2 = -1$ are scalar multiples of

$$\begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$$

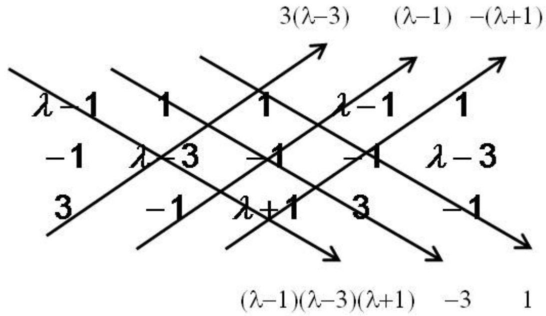
- **Example:** Find the eigenvalues and eigenvectors for

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix}$$

To find the eigenvalues:

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & 1 & 1 \\ -1 & \lambda - 3 & -1 \\ 3 & -1 & \lambda + 1 \end{vmatrix}$$

Using the permutation approach:



$$\begin{aligned}
|\lambda I - A| &= (\lambda-1)(\lambda-3)(\lambda+1) - 2 - [3(\lambda-3) + (\lambda-1) - (\lambda+1)] \\
&= (\lambda^2 - 4\lambda + 3)(\lambda+1) - 2 - [3\lambda - 9 + \lambda - 1 - \lambda - 1] \\
&= \lambda^3 - 3\lambda^2 - 4\lambda + 12 \\
&= \lambda^2(\lambda-3) - 4(\lambda-3) \\
&= (\lambda-3)(\lambda^2-4) \\
&= (\lambda-3)(\lambda-2)(\lambda+2)
\end{aligned}$$

Therefore, to find the eigenvalues we need to solve $|\lambda I - A| = 0$ which is

$$(\lambda-3)(\lambda-2)(\lambda+2) = 0.$$

Therefore, the three eigenvalues for A are $\lambda_1 = 3$, $\lambda_2 = 2$, and $\lambda_3 = -2$. Finding the eigenvector for $\lambda_1 = 3$: We need to solve $(\lambda_1 I - A)x = 0$ for x. Substituting λ_1 into the form above for $\lambda I - A$, we have the system

$$\left[\begin{array}{rrr|r} 2 & 1 & 1 & 0 \\ -1 & 0 & -1 & 0 \\ 3 & -1 & 4 & 0 \end{array}\right] \rightarrow_{R_1 \leftrightarrow R_2} \left[\begin{array}{rrr|r} -1 & 0 & -1 & 0 \\ 2 & 1 & 1 & 0 \\ 3 & -1 & 4 & 0 \end{array}\right]$$

$$\rightarrow_{R_1 \leftrightarrow -R_1} \left[\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 \\ 3 & -1 & 4 & 0 \end{array}\right]$$

$$\rightarrow_{\substack{R_2 \leftrightarrow R_2 - 2R_1 \\ R_3 \leftrightarrow R_3 - 3R_1}} \left[\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right]$$

$$\rightarrow_{R_3 \leftrightarrow R_3 + R_2} \left[\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So, $x_3 = t$, $x_2 = x_3 = t$, and $x_1 = -x_3 = -t$. (Remember, there should ALWAYS be a free variable when finding the eigenvector.) So, the eigenvector for $\lambda_1 = 3$ is any scalar multiple of

$$\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Finding the eigenvector for $\lambda_2 = 2$: We need to solve $(\lambda_2 I - A)x = 0$ for x. Substituting λ_2 into the form above for $\lambda I - A$, we have the system

$$\begin{bmatrix} 1 & 1 & 1 & | & 0 \\ -1 & -1 & -1 & | & 0 \\ 3 & -1 & 3 & | & 0 \end{bmatrix} \rightarrow_{\substack{R_2 \leftrightarrow R_2 + R_1 \\ R_3 \leftrightarrow R_3 - 3R_1}} \begin{bmatrix} 1 & 1 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & -4 & 0 & | & 0 \end{bmatrix}$$

$$\rightarrow_{\substack{R_3 \leftrightarrow -\frac{1}{4}R_3 \\ R_3 \leftrightarrow R_2}} \begin{bmatrix} 1 & 1 & 1 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\rightarrow_{R_1 \leftrightarrow R_1 - R_2} \begin{bmatrix} 1 & 0 & 1 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

So, $x_3 = t$, $x_2 = 0$, and $x_1 = -x_3 = -t$. So, the eigenvector for $\lambda_2 = 2$ is any scalar multiple of

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Finding the eigenvector for $\lambda_3 = -2$: We need to solve $(\lambda_3 I - A)x = 0$ for x. Substituting λ_3 into the form above for $\lambda I - A$, we have the system

$$\begin{bmatrix} -3 & 1 & 1 & | & 0 \\ -1 & -5 & -1 & | & 0 \\ 3 & -1 & -1 & | & 0 \end{bmatrix} \rightarrow_{R_2 \leftrightarrow R_1} \begin{bmatrix} -1 & -5 & -1 & | & 0 \\ -3 & 1 & 1 & | & 0 \\ 3 & -1 & -1 & | & 0 \end{bmatrix}$$

$$\rightarrow_{R_1 \leftrightarrow -R_1} \begin{bmatrix} 1 & 5 & 1 & | & 0 \\ -3 & 1 & 1 & | & 0 \\ 3 & -1 & -1 & | & 0 \end{bmatrix}$$

$$\rightarrow_{R_3 \leftrightarrow R_3 + R_2} \begin{bmatrix} 1 & 5 & 1 & | & 0 \\ -3 & 1 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\rightarrow_{R_2 \leftrightarrow R_2 + 3R_1} \begin{bmatrix} 1 & 5 & 1 & | & 0 \\ 0 & -16 & -4 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\rightarrow_{R_2 \leftrightarrow -1/16R_2} \begin{bmatrix} 1 & 5 & 1 & | & 0 \\ 0 & 1 & \frac{1}{4} & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\rightarrow_{R_1 \leftrightarrow -5R_2} \begin{bmatrix} 1 & 0 & -\frac{1}{4} & | & 0 \\ 0 & 1 & \frac{1}{4} & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

So, $x_3 = t$, $x_2 = -1/4x_3 = -1/4t$, and $x_1 = 1/4x_3 = 1/4t$. So, the eigenvector for $\lambda_3 = -2$ is any scalar multiple of

$$\begin{bmatrix} 1/4 \\ -1/4 \\ 1 \end{bmatrix}$$

- **Example:** Find the eigenvalues and corresponding eigenvectors for

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

Answer: eigenvalue $\lambda_1 = 2$ with eigenvectors

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

and eigenvalue $\lambda_2 = 1$ with eigenvector

$$\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

- **Eigenvalues of Upper Triangular, Lower Triangular, and Diagonal Matrices:** If A is an nxn triangular matrix (upper triangular, lower triangular, or diagonal), then the eigenvalues of A are the entries on the main diagonal of A

  - **Example:** Using the matrix A from above,

    $$A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix}$$

    We can easily pick off the eigenvalues of A since it is upper triangular. Thus, the eigenvalues are the elements on the diagonal of A, i.e. 1, 1, and 2. So, A only has 2 distinct eigenvalues.

- **Powers of a Matrix:** If k is a positive integer, λ is an eigenvalue of a matrix A, and $\mathbf{x}$ is a corresponding eigenvector, then λ^k is an eigenvalue of A^k and $\mathbf{x}$ is a corresponding eigenvector.

  - **Example:** From above we have that $\lambda = 2$ and $\lambda = 1$ are eigenvalues of

    $$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

    Therefore, both $\lambda = 2^7 = 128$ and $\lambda = 1^7 = 1$ are eigenvalues of A^7 with

    $$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

    as corresponding eigenvectors for $\lambda = 2^7 = 128$ (the same eigenvectors for $\lambda = 2$ from A and

    $$\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

    as the cooresponding eigenvector for $\lambda = 1^7 = 1$.

- **Eigenvalues and Invertibility:** A square matrix A is invertible if and only if $\lambda = 0$ is *not* and eigenvalue of A.

- **Equivalent Statements:** If A is an nxn matrix, then the following are equivalent:

  (a) A is invertible
  (b) $Ax = 0$ has only the trivial solution
  (c) The reduced row-echelon form of A is I_n
  (d) A is expressible as a product of elementary matrices
  (e) $Ax = b$ is consistent for every nx1 matrix b
  (f) $Ax = b$ has exactly one solution for every nx1 matrix b
  (g) $|A| \neq 0$
  (h) $\lambda = 0$ is not an eigenvalue of A