Elementary Matrices
Linear Algebra
MATH 2010

- Recall that there are three *elementary row operations* that can be performed on a system and still have an equivalent system:
  1. interchange two rows
  2. multiply a row by a nonzero constant
  3. add a multiple of one row to another row
- If any of these three operations are performed on a matrix A to obtain a matrix B, then matrices A and B are said to be **row equivalent**.
- Matrix multiplication can also be used to carry out the elementary row operation.
- **Elementary Matrix:** An nxn matrix is called an *elementary matrix* if it can be obtained from the nxn identity I_n by performing a single elementary row operation.
- **Examples:**
  - $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Elementary operation performed: multiply second row by $-\frac{1}{3}$.
  - $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ Elementary operation performed: interchanging rows 2 and 4.
  - $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$ Elementary operation performed: adding -2 times the first row to the third row.
- **Use of Elementary Matrices:** Let A be a mxn matrix and let E be an mxm elementary matrix. Then the matrix multiplication EA is the matrix that results when the same row operation is performed on A as that performed to produce the elementary matrix E. For example:

$$A = \begin{bmatrix} 1 & -4 & 3 \\ 0 & 2 & 0 \\ -1 & 3 & 7 \end{bmatrix}$$

The matrix

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is the matrix found by adding -1 times the first row to the second row. Multiply E_1A, you get

$$E_1A = \begin{bmatrix} 1 & -4 & 3 \\ -1 & 6 & -3 \\ -1 & 3 & 7 \end{bmatrix}$$

which is the same as if you had added -1 times row 1 of A to row 2 of A.

- **Example:** Let

$$A = \begin{bmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 8 & 1 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 8 & 1 & 5 \\ 2 & -7 & -1 \\ 3 & 4 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 2 & -7 & 3 \end{bmatrix}$$

  Find elementary matrices E_1, E_2, E_3, and E_4 such that

  1. $E_1 A = B$
  2. $E_2 B = A$
  3. $E_3 A = C$
  4. $E_4 C = A$

- **Row reduction:** The row reduction of a matrix A to row-echelon form H can be accomplished by successive multiplication on the left by elementary matrices:

$$H = (E_r E_{r-1} \cdots E_2 E_1)A.$$

- **Example:** Find a matrix C such that CA is a matrix in row-echelon form that is row equivalen to A where C is a product of elementary matrices. We will consider the example from the Linear Systems section where

$$A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 1 & 3 & 0 & 5 \\ 2 & 7 & 2 & 9 \end{bmatrix}$$

  So, begin with row reduction:

Original matrix	Elementary row operation	Resulting matrix	Associated elementary matrix
$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 1 & 3 & 0 & 5 \\ 2 & 7 & 2 & 9 \end{bmatrix}$	$\rightarrow_{R_2 \leftrightarrow R_2 + -1R_1}$	$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 1 & 1 \\ 2 & 7 & 2 & 9 \end{bmatrix}$	$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
	$\rightarrow_{R_3 \leftrightarrow R_3 + -2R_1}$	$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & 4 & 1 \end{bmatrix}$	$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$
	$\rightarrow_{R_3 \leftrightarrow R_3 + -2R_2}$	$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$	$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

  Then

$$C = E_3 E_2 E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & -2 & 1 \end{bmatrix}$$

- Obviously, elementary matrices are not the most efficient way to perform Gaussian elemination. We will discuss elementary matrices a little more when we talk about *inverses* of a matrix.