Inverses
Linear Algebra
MATH 2010

- For matrix addition, the matrix O is similar to the zero for real numbers, because

$$A + O = A;$$
$$A - A = O$$

- For matrix multiplication, I is similar to the scalar 1 in real numbers, because

$$IA = AI = A$$

- For scalar multiplication, however, there is also a reciprocal. For example,

$$2 \cdot \frac{1}{2} = 1$$

For all real numbers c, we have

$$c \cdot \frac{1}{c} = 1$$

We want to have a similar idea for matrices, we want a matrix such that when we multiply it with A, we get the identity I.

- **Definition:** A nxn matrix A is **invertible** (or **nonsingular**) if there exists a nxn matrix B such that $AB = BA = I_n$. B is called the **inverse** of A and is denoted by A^{-1}.
- If the matrix A does not have an inverse, then A is said to be **noninvertible** (or **nonsingular**).
- **Example:** Show that

$$B = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

is the inverse of

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

In order to show the matrix B is the inverse of A, we need to show that both $AB = I$ and $BA = I$.

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$BA = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- **Inverse for a 2x2 matrix:** If A is 2x2, i.e.,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then A is invertible if and only if $ad - bc \neq 0$. In this case the inverse is given by

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- **Example:** Let
$$A = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$$
Find A^{-1} if it exists.

The first thing to check is whether or not A^{-1} exists. Since it is a 2x2 matrix, we know A^{-1} exists if and only if $ad - bc \neq 0$.
$$ad - bc = 2(3) - 0(1) = 6 \neq 0$$
so A^{-1} exists and is given by
$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{6} & \frac{1}{3} \end{bmatrix}$$

- **Finding A^{-1}:**
  - To find A^{-1} for a general nxn matrix A, we want to find the matrix $X = A^{-1}$ such that
$$AX = I (\text{or } XA = I)$$
Let's first examine the 2x2 case in which we already know the form of the inverse. For example for
$$A = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}$$
we want to find a matrix
$$X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$$
such that
$$AX = I$$
or
$$\begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
If we multiply the left side out and set it equal to the right side, we get the systems
$$\begin{array}{rcrcr} -x_{11} & + & 2x_{12} & = & 1 \\ -x_{11} & + & x_{21} & = & 0 \end{array} \qquad \begin{array}{rcrcr} -x_{12} & + & 2x_{22} & = & 0 \\ -x_{12} & + & x_{22} & = & 1 \end{array}$$
Notice, that both systems have the same coefficient matrix:
$$\left[\begin{array}{rr|r} -1 & 2 & 1 \\ -1 & 1 & 0 \end{array}\right] \qquad \left[\begin{array}{rr|r} -1 & 2 & 0 \\ -1 & 1 & 1 \end{array}\right]$$
So, we can solve the systems simultaneously:
$$\left[\begin{array}{rr|rr} -1 & 2 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{array}\right]$$
This is the same as solving
$$[A|I]$$
We can solve until we have reduced row echelon form on the left, then $X = A^{-1}$ will be on the right. In other words, if there is a unique solution, we can reduce so that we have
$$[A|I] \rightarrow \left[I|A^{-1}\right].$$
  - We can see this process a different way by looking at elementary matrices. If
$$E_k E_{k-1} \cdots E_2 E_1 A = I_n,$$
then if A is invertible,
$$A^{-1} = E_k E_{k-1} \cdots E_2 E_1 I_n$$
by multiplying on the right by A^{-1} on both sides of the equation. This show exactly what we discussed previously, the same sequence of row operations that reduces A to I_n will transform I_n to A^{-1}.

- **Example:** Let's do this with the matrix above:

$$\left[\begin{array}{cc|cc} -1 & 2 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{array}\right] \rightarrow_{R_1 \leftrightarrow -R_1} \left[\begin{array}{cc|cc} 1 & -2 & -1 & 0 \\ -1 & 1 & 0 & 1 \end{array}\right]$$

$$\rightarrow_{R_2 \leftrightarrow R_2 + R_1} \left[\begin{array}{cc|cc} 1 & -2 & -1 & 0 \\ 0 & -1 & -1 & 1 \end{array}\right]$$

$$\rightarrow_{R_2 \leftrightarrow -R_2} \left[\begin{array}{cc|cc} 1 & -2 & -1 & 0 \\ 0 & 1 & 1 & -1 \end{array}\right]$$

$$\rightarrow_{R_1 \leftrightarrow R_1 + 2R_2} \left[\begin{array}{cc|cc} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & -1 \end{array}\right]$$

Thus,

$$A^{-1} = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$$

You can check that $A \cdot A^{-1} = I$. Using the 2x2 matrix way, we would have had

$$A^{-1} = \frac{1}{-1(1) - 2(-1)} \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} = \frac{1}{1} \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$$

This is the same as above.

- **Example:** Find the inverse, if it exists for the matrix

$$A = \begin{bmatrix} 3 & 2 & 5 \\ 2 & 2 & 4 \\ -4 & 4 & 0 \end{bmatrix}$$

So, set up the augmented system $[A|I]$ and reduce to $[I|A^{-1}]$.

$$\left[\begin{array}{ccc|ccc} 3 & 2 & 5 & 1 & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ -4 & 4 & 0 & 0 & 0 & 1 \end{array}\right] \overset{R_1 \leftrightarrow R_3 + R_3}{\longrightarrow} \left[\begin{array}{ccc|ccc} -1 & 6 & 5 & 1 & 0 & 1 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ -4 & 4 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\overset{R_1 \leftrightarrow -R_1}{\longrightarrow} \left[\begin{array}{ccc|ccc} 1 & -6 & -5 & -1 & 0 & -1 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ -4 & 4 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$R_2 \leftrightarrow R_2 + -2R_1 \overset{\text{and}}{\longrightarrow} R_3 \leftrightarrow R_3 + 4R_1 \left[\begin{array}{ccc|ccc} 1 & -6 & -5 & -1 & 0 & -1 \\ 0 & 14 & 14 & 2 & 1 & 2 \\ 0 & -20 & -20 & -4 & 0 & -3 \end{array}\right]$$

$$\overset{R_2 \leftrightarrow \frac{1}{14} R_2}{\longrightarrow} \left[\begin{array}{ccc|ccc} 1 & -6 & -5 & -1 & 0 & -1 \\ 0 & 1 & 1 & \frac{1}{7} & \frac{1}{14} & \frac{1}{7} \\ 0 & -20 & -20 & -4 & 0 & -3 \end{array}\right]$$

$$\overset{R_3 \leftrightarrow R_3 + 20R_2}{\longrightarrow} \left[\begin{array}{ccc|ccc} 1 & -6 & -5 & -1 & 0 & -1 \\ 0 & 1 & 1 & \frac{1}{7} & \frac{1}{14} & \frac{1}{7} \\ 0 & 0 & 0 & -\frac{8}{7} & \frac{10}{7} & -\frac{1}{7} \end{array}\right]$$

Since, the bottom row has all zeros to the left and nonzero to the right, it is impossible to reduce the A to I, hence A^{-1} does not exist. A is singular.

- Try to find A^{-1} given

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix}$$

- **Properties of A^{-1}:**
  - $(A^{-1})^{-1} = A$
  - $(A^k)^{-1} = (A^{-1})^k$
  - $(cA)^{-1} = \frac{1}{c}A^{-1}$, $c \neq 0$
  - $(A^T)^{-1} = (A^{-1})^T$
  - $(AB)^{-1} = B^{-1}A^{-1}$
- **Example:** Given

$$A^{-1} = \begin{bmatrix} 2 & 5 \\ -7 & 6 \end{bmatrix}$$

  find the following without computing A:

  1. $(A^T)^{-1}$; Ans: $\begin{bmatrix} 2 & -7 \\ 5 & 6 \end{bmatrix}$
  2. A^{-2}; Ans: $\begin{bmatrix} -31 & 40 \\ -56 & 1 \end{bmatrix}$
  3. $(2A)^{-1}$; Ans: $\begin{bmatrix} 1 & \frac{5}{2} \\ -\frac{7}{2} & 3 \end{bmatrix}$
  4. Given $B^{-1} = \begin{bmatrix} 7 & -3 \\ 2 & 0 \end{bmatrix}$, find $(AB)^{-1}$; Ans: $\begin{bmatrix} 35 & 17 \\ 4 & 10 \end{bmatrix}$

- **Cancellation:** If C is invertible, then
  - If $AC = BC$, then $A = B$.

    Proof:

    $AC = BC$

    $ACC^{-1} = BCC^{-1}$ since, C^{-1} exists, we can multiply the equation on the right by C^{-1}.

    $AI = BI$ because by definition of inverse, $CC^{-1} = I$

    $A = B$ because $AI = A$ for any matrix A by property of the identity matrix.
  - If $CA = CB$, then $A = B$. (similar proof)

- **Systems of Equations:** Assume you have the system $Ax = b$, then if A is invertible, there is a unique solution to the system given by

$$x = A^{-1}b.$$

  Proof:

  $Ax = b$

  $A^{-1}Ax = A^{-1}b$ since, A^{-1} exists, we can multiply the equation on the left by A^{-1}.

  $Ix = A^{-1}b$ because by definition of inverse, $AA^{-1} = I$

  $x = A^{-1}b$ because $IB = B$ for any matrix B by property of the identity matrix.

- **Example:** Solve the following system using an inverse matrix.

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & - & 2x_3 & = & 0 \\ x_1 & - & 2x_2 & + & x_3 & = & 0 \\ x_1 & - & x_2 & - & x_3 & = & -1 \end{array}$$

  Using

$$A = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -2 & 1 \\ 1 & -1 & -1 \end{bmatrix}$$

we can find that A^{-1} exists and is given by

$$A^{-1} = \begin{bmatrix} 2/3 & 1/3 & 0 \\ 1/3 & -1/3 & 0 \\ -1/3 & -2/3 & 1 \end{bmatrix}$$

Thus, the unique solution to the system is given by

$$x = A^{-1}b = \begin{bmatrix} 2/3 & 1/3 & 0 \\ 1/3 & -1/3 & 0 \\ -1/3 & -2/3 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- **Invertibility of Elementary Matrices:** Every elementary matrix is invertible, and the inverse is also an elementary matrix.
- **Equivalent Statements:** If A is an nxn matrix, then the following are equivalent:
  (a) A is invertible
  (b) $Ax = 0$ has only the trivial solution
  (c) The reduced row-echelon form of A is I_n
  (d) A is expressible as a product of elementary matrices.
  (e) $Ax = b$ is consistent for every nx1 matrix b
  (f) $Ax = b$ has exactly one solution for every nx1 matrix b