Linear Systems
Linear Algebra
MATH 2010

- **Overview of a Linear Equation and System of Linear Equations** In general a *linear equation* in n variables, namely x_1, x_2, ..., x_n has the form

$$a_1x_1 + a_2x_2 + a_3x_3 + ... + a_nx_n = b$$

  where a_1, a_2, ..., a_n and b are all constant real numbers. All variables have an exponent of 1 with a simple constant multiplying the variable. A *system of linear equations* is a set of 2 or more linear equations.

- **Categories of Possible Solutions** of Linear Equations:

  1. **Unique Solution:** There is a unique point $(x_1, x_2, ..., x_n)$ to the system of linear equations. For example, the system

$$\begin{array}{rcrcl} x & + & y & = & 2 \\ 3x & - & y & = & 0 \end{array}$$

     is a linear system in two variables with a unique solution where the two lines intersect at the point $(\frac{1}{2}, \frac{3}{2})$. We can visualize this in Euclidean 2-space for a general linear system of two equation where the two lines cross as a unique point:
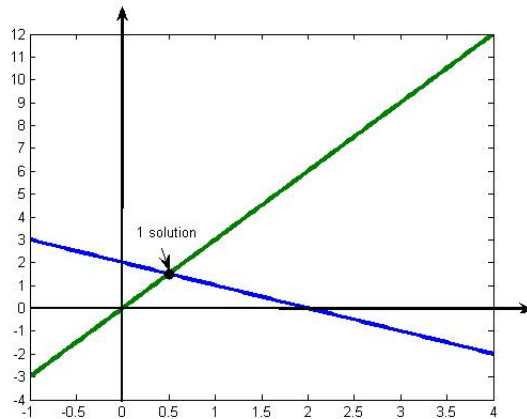
  2. **No Solution:** There is no point $(x_1, x_2, ..., x_n)$ which satisfy ALL the linear equations in the system. For example, the system

$$\begin{array}{rcrcl} x & + & y & = & 4 \\ x & + & y & = & 3 \end{array}$$

     is a linear system in two variables where the two lines are parallel (never cross). We can visualize this in Euclidean 2-space for a general linear system of two equation where the two lines are parallel with the same slope but different y-intercept:
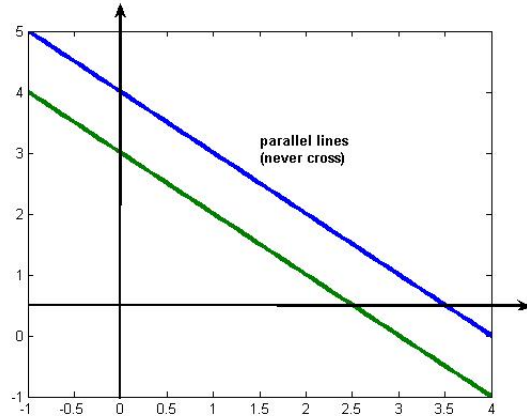
3. **Infinitely Many Solutions:** There is are infinitely many points $(x_1, x_2, ..., x_n)$ which satisfy ALL the linear equations in the system. For example, the system

$$\begin{array}{ccccc} x & + & y & = & 2 \\ 2x & + & 2y & = & 4 \end{array}$$

is a linear system in two variables where the two lines lie on top of each other. We can visualize this in Euclidean 2-space for a general linear system of two equation where one equation is a multiple of the other equation:
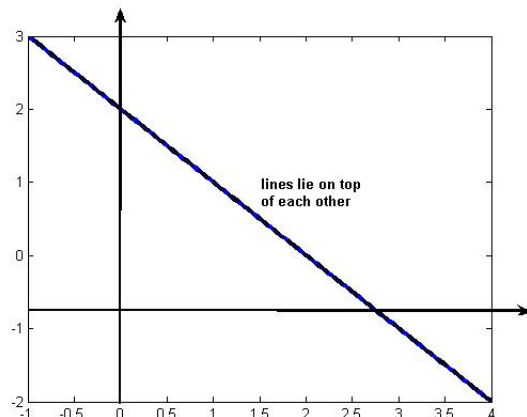
- **Consistent vs. Inconsistent:**

    - **Consistent:** If a system of equations has *at least one* solution, then the system is called *consistent*.
    - **Inconsistent:** If there are no solutions to the system of equations, then the system is called *inconsistent*.

- **Solution Set:** The set of all solutions is called the *solution set*.

- **Parametric Representation for Infinite Solutions:** Consider the linear system of equations

$$\begin{array}{ccccc} 3x & - & 4y & = & 1 \\ 6x & - & 8y & = & 2 \end{array}$$

The second equation is a multiple of the first equation, i.e., if you multiply the first equation $3x - 4y = 1$ by 2, you get the second equation $6x - 8y = 2$. So, the solution set for the system is the same as the solution set for the equation $3x - 4 = 1$. We represent an *infinite* solution set using *parametric representation*. To find the parametric representation in a two-variable system,

1. Assign one of the variables, typically the second variable in the equation, a parameter value such as t or s, where the parameter can take on any real number. In this example, we can assign

$$y = t,$$

   so the equation becomes $3x - 4t = 1$.

2. Solve the equation for the other variable:

$$3x - 4t = 1 \rightarrow 3x = 1 + 4t \rightarrow x = \frac{1}{3} + \frac{4}{3}t.$$

3. The solution set consists of all the points (x, y) to the system. Therefore, in this example, the solution set is given by

$$\left(\frac{1}{3} + \frac{4}{3}t, t\right)$$

   where t is any real number.

A *particular solution* can be found by setting t to some real number. For example, if we let $t = -1$, then

$$\left(\frac{1}{3} + \frac{4}{3}t, t\right) = \left(\frac{1}{3} + \frac{4}{3}(-1), -1\right) = (-1, -1)$$

is a particular solution.

- **Equivalent System:** Now let's find the solution sets to more complicated systems. Consider the system

$$\begin{array}{ccccccc}
x & + & 2y & - & z & = & 4 \\
x & + & 3y & & & = & 5 \\
2x & + & 7y & + & 2z & = & 9
\end{array}$$

It is difficult to find the solution to this system in the given form. However, this system can be reduced to the equivalent system

$$\begin{array}{ccccccc}
x & + & 2y & - & z & = & 4 \\
& & y & + & z & = & 1 \\
& & & & z & = & -2
\end{array}$$

The latter system can easily be solved using *back-substitution*:

$$z = -2 \rightarrow y = 1 - z = 1 - (-2) = 3 \rightarrow x = 4 - 2y + z = 4 - 2(3) + (-2) = -4.$$

There are three *elementary row operations* that can be performed on a system and still have an equivalent system:

1. interchange two equations
2. multiply an equation by a nonzero constant
3. add a multiple of one equation to another equation

- Note that the solution can be checked by plugging in the answer into the *original* system.

- **Augmented System:** Let's consider the system above again:

$$\begin{array}{ccccccc}
x & + & 2y & - & z & = & 4 \\
x & + & 3y & & & = & 5 \\
2x & + & 7y & + & 2z & = & 9
\end{array}$$

It can be written using only the coefficients by using matrices.

– **Basic Definition and Notation for Matrices**

* If m and n are positive integers, then an mxn **matrix** is a rectangular array of numbers (entries)

$$m \text{ rows} \begin{cases} \begin{bmatrix} a_{11} & a_{12} & a_{13} & ... & a_{1n} \\ a_{21} & a_{22} & a_{23} & ... & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & ... & a_{mn} \end{bmatrix} \end{cases}$$

$$\underbrace{}_{n \text{ columns}}$$

where a_{ij} is the number corresponding to the ith row and j^{th} column. i is the row subscript and j is the column subscript.

* The **size** of the matrix is mxn.

* Matrices are denoted by capital letters: A, B, C, etc.

– **Back to System:** The above system can as an augmented system by eliminating the variables and simply using the coefficients:

$$\begin{bmatrix} 1 & 2 & -1 & | & 4 \\ 1 & 3 & 0 & | & 5 \\ 2 & 7 & 2 & | & 9 \end{bmatrix}$$

The equivalent system

$$\begin{array}{ccccccc} x & + & 2y & - & z & = & 4 \\ & & y & + & z & = & 1 \\ & & & & z & = & -2 \end{array}$$

is written in augmented form by

$$\begin{bmatrix} 1 & 2 & -1 & | & 4 \\ 0 & 1 & 1 & | & 1 \\ 0 & 0 & 1 & | & -2 \end{bmatrix}$$

The latter augmented form is in **row-echelon form**.

• **Row-echelon form:** In general, a matrix is in row-echelon form if it has the following properties:

1. All row consisting of entirely zeros occur at the bottom of the matrix.

2. For each row that does not consist of entirely zeros, the first nonzero entry is 1 (this is called the leading 1 or *pivot*).

3. For two successive (nonzero) rows, the leading 1 in the higher row is farther to the left than the leading 1 in the lower row.

Which of the following matrices are in row-echelon form?

1. $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ Yes.

2. $\begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ Yes.

3. $\begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ No.

4. $\begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -4 \end{bmatrix}$ No.

5. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ No.

- **Gaussian Elimination:** Gaussian elimination is the process of reducing a system to row-echelon form by performing the elementary row operations described above:

  1. interchange two equations (or two rows in augmented form); typical notation for interchanging row i and row j: $R_i \leftrightarrow R_j$.

  2. multiply an equation by a nonzero constant (or multiply a row by a constant); typical notation for multiplying row i by a nonzero constant c: $R_i \leftrightarrow cR_i$.

  3. add a multiple of one equation to another equation (or add a multiple of one row to another row); typical notation for adding c times row j to row i: $R_i \leftrightarrow R_i + cR_j$.

  Let's work through our example using the elementary row operations.

$$\begin{array}{rcrcrcl}
x & + & 2y & - & z & = & 4 \\
x & + & 3y & & & = & 5 \\
2x & + & 7y & + & 2z & = & 9
\end{array}$$

  The associated augmented form is

$$\left[\begin{array}{ccc|c}
1 & 2 & -1 & 4 \\
1 & 3 & 0 & 5 \\
2 & 7 & 2 & 9
\end{array}\right]$$

  Starting with the augmented form, use the elementary operations to find the row-echelon form:

$$\left[\begin{array}{ccc|c}
1 & 2 & -1 & 4 \\
1 & 3 & 0 & 5 \\
2 & 7 & 2 & 9
\end{array}\right] \xrightarrow{R_2 \leftrightarrow R_2 + -1R_1}
\left[\begin{array}{ccc|c}
1 & 2 & -1 & 4 \\
0 & 1 & 1 & 1 \\
2 & 7 & 2 & 9
\end{array}\right]$$

$$\xrightarrow{R_3 \leftrightarrow R_3 + -2R_1}
\left[\begin{array}{ccc|c}
1 & 2 & -1 & 4 \\
0 & 1 & 1 & 1 \\
0 & 3 & 4 & 1
\end{array}\right]$$

$$\xrightarrow{R_3 \leftrightarrow R_3 + -3R_2}
\left[\begin{array}{ccc|c}
1 & 2 & -1 & 4 \\
0 & 1 & 1 & 1 \\
0 & 0 & 1 & -2
\end{array}\right]$$

  Then the corresponding system is

$$\begin{array}{rcrcrcr}
x & + & 2y & - & z & = & 4 \\
& & y & + & z & = & 1 \\
& & & & z & = & -2
\end{array}$$

  with solution $(-4, 3, -2)$.

- **Example:** Consider the system

$$\begin{array}{rcrcrcrcrcrcr}
x_1 & + & 3x_2 & - & 2x_3 & & & + & 2x_5 & & & = & 0 \\
2x_1 & + & 6x_2 & - & 5x_3 & - & 2x_4 & + & 4x_5 & - & 3x_6 & = & -1 \\
& & & & 5x_3 & + & 10x_4 & & & + & 15x_6 & = & 5 \\
2x_1 & + & 6x_2 & & & + & 8x_4 & + & 4x_5 & + & 18x_6 & = & 6
\end{array}$$

  Augmented Form:

$$\left[\begin{array}{cccccc|c}
1 & 3 & -2 & 0 & 2 & 0 & 0 \\
2 & 6 & -5 & -2 & 4 & -3 & -1 \\
0 & 0 & 5 & 10 & 0 & 15 & 5 \\
2 & 6 & 0 & 8 & 4 & 18 & 6
\end{array}\right]$$

Let's reduce the following system:

$$
\left[\begin{array}{cccccc|c}
1 & 3 & -2 & 0 & 2 & 0 & 0 \\
2 & 6 & -5 & -2 & 4 & -3 & -1 \\
0 & 0 & 5 & 10 & 0 & 15 & 5 \\
2 & 6 & 0 & 8 & 4 & 18 & 6
\end{array}\right]
\quad
\underset{\longrightarrow}{R_2 \leftrightarrow R_2 - 2R_1 \text{ and } R_4 \leftrightarrow R_4 - 2R_1}
\quad
\left[\begin{array}{cccccc|c}
1 & 3 & -2 & 0 & 2 & 0 & 0 \\
0 & 0 & -1 & -2 & 0 & -3 & -1 \\
0 & 0 & 5 & 10 & 0 & 15 & 5 \\
0 & 0 & 4 & 8 & 0 & 18 & 6
\end{array}\right]
$$

$$
\underset{\longrightarrow}{R_2 \leftrightarrow -R_2}
\quad
\left[\begin{array}{cccccc|c}
1 & 3 & -2 & 0 & 2 & 0 & 0 \\
0 & 0 & 1 & 2 & 0 & 3 & 1 \\
0 & 0 & 5 & 10 & 0 & 15 & 5 \\
0 & 0 & 4 & 8 & 0 & 18 & 6
\end{array}\right]
$$

$$
\underset{\longrightarrow}{R_3 \leftrightarrow R_3 - 5R_2 \text{ and } R_4 \leftrightarrow R_4 - 4R_2}
\quad
\left[\begin{array}{cccccc|c}
1 & 3 & -2 & 0 & 2 & 0 & 0 \\
0 & 0 & 1 & 2 & 0 & 3 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 6 & 2
\end{array}\right]
$$

$$
\underset{\longrightarrow}{R_4 \leftrightarrow \frac{1}{6} R_4 \text{ and } R_3 \leftrightarrow R_4}
\quad
\left[\begin{array}{cccccc|c}
1 & 3 & -2 & 0 & 2 & 0 & 0 \\
0 & 0 & 1 & 2 & 0 & 3 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
$$

Now, to write down the solution, we need to determine which variables are free parameters and which are dependent variables. Columns without a leading 1 or pivot correspond to free variables (see the figure below).

So, let

$$
\begin{aligned}
x_2 &= r \\
x_4 &= s \\
x_5 &= t
\end{aligned}
$$

Now, solve the systems for the remaining variables from the last equation to the first equation:

$$
\begin{aligned}
x_1 + 3x_2 - 2x_3 \qquad\quad + 2x_5 \qquad\qquad\quad &= 0 \\
x_3 + 2x_4 + \qquad\quad + 3x_6 &= 1 \\
+ \quad x_6 &= \tfrac{1}{3}
\end{aligned}
$$

Plugging in the parameters, we have

$$
\begin{aligned}
x_1 + 3r - 2x_3 \qquad\quad + 2t \qquad\qquad\quad &= 0 \\
x_3 + 2s + \qquad\quad + 3x_6 &= 1 \\
+ \quad x_6 &= \tfrac{1}{3}
\end{aligned}
$$

Then,

$$
x_6 = \frac{1}{3}
$$

and
$$x_3 + 2s + 3x_6 = 1 \rightarrow x_3 = 1 - 2s - 3\left(\frac{1}{3}\right) \rightarrow x_3 = -2s$$

and

$$x_1 + 3r - 2x_3 + 2t = 0 \rightarrow x_1 = -3r + 2x_3 - 2t \rightarrow x_1 = -3r + 2(-2s) - 2t \rightarrow x_1 = -3r - 4s - 2t.$$

Then the solution is given by:
$$\left(-3r - 4s - 2t, r, -2s, s, t, \frac{1}{3}\right).$$

- **Example:** Consider the simple system:

$$\begin{array}{rrrrr} x & - & y & = & 1 \\ -2x & + & 2y & = & 5 \end{array}$$

Reducing this system to row echelon form, we get

$$\left[\begin{array}{rr|r} 1 & -1 & 1 \\ -2 & 2 & 5 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_2 + 2R_1} \left[\begin{array}{rr|r} 1 & -1 & 1 \\ 0 & 0 & 7 \end{array}\right]$$

This gives the system

$$\begin{array}{rrrrr} x & - & y & = & 1 \\ & & 0 & = & 7 \end{array}$$

This is not possible, so there is NO solution!!

- **Problems:** Solve the following problems using Gaussian Elimination

1.
$$\begin{array}{rrrrr} x & + & 2y & = & 7 \\ 2x & + & y & = & 8 \end{array}$$
   Ans: (3,2)

2.
$$\begin{array}{rrrrr} -3x & + & 5y & = & -22 \\ 3x & + & 4y & = & 4 \\ 4x & - & 8y & = & 32 \end{array}$$
   Ans: (4,-2)

3.
$$\begin{array}{rrrrr} -x & + & 2y & = & \frac{3}{2} \\ 2x & - & 4y & = & 3 \end{array}$$
   Ans: No solution

4.
$$\begin{array}{rrrrrrr} x_1 & + & x_2 & - & 5x_3 & = & 3 \\ x_1 & & & - & 2x_3 & = & 1 \\ 2x_1 & - & x_2 & - & x_3 & = & 0 \end{array}$$
   Ans: $(1 + 2t, 2 + 3t, t)$

5.
$$\begin{array}{rrrrrrr} 3x_1 & + & 3x_2 & + & 12x_3 & = & 6 \\ x_1 & + & x_2 & + & 4x_3 & = & 2 \\ 2x_1 & + & 5x_2 & + & 20x_3 & = & 10 \\ -x_1 & + & 2x_2 & + & 8x_3 & = & 4 \end{array}$$
   Ans: $(0, 2 - 4t, t)$

- **Gauss-Jordan Reduction:** Gauss-Jordan reduction reduces the system to *reduced row-echelon form*. Reduced-echelon form has zeros above and below the leading ones (as opposed to simply below the leading ones). Let's look at the following example:

$$
\begin{array}{rrrcr}
2x_1 & - & x_2 & + & 3x_3 & = & 24 \\
 & & 2x_2 & - & x_3 & = & 14 \\
3x_1 & & & & & = & 6
\end{array}
$$

The augmented form is:

$$
\left[\begin{array}{ccc|c}
2 & -1 & 3 & 24 \\
0 & 2 & -1 & 14 \\
3 & 0 & 0 & 6
\end{array}\right]
$$

Notice that if we simply multiply the first equation by $\frac{1}{2}$ to make the leading number in the first row a 1, you will encounter multiple fractions in the first row. You can perform any elementary row operation in any order. So, we can instead subtract row 3 and row 1 first:

$$
\left[\begin{array}{ccc|c}
2 & -1 & 3 & 24 \\
0 & 2 & -1 & 14 \\
3 & 0 & 0 & 6
\end{array}\right]
\quad \overset{R_3 \leftrightarrow R_3 - R_2}{\longrightarrow} \quad
\left[\begin{array}{ccc|c}
2 & -1 & 3 & 24 \\
0 & 2 & -1 & 14 \\
1 & 1 & -3 & -18
\end{array}\right]
$$

$$
\overset{R_2 \leftrightarrow R_3}{\longrightarrow}
\left[\begin{array}{ccc|c}
1 & 1 & -3 & -18 \\
0 & 2 & -1 & 14 \\
2 & -1 & 3 & 24
\end{array}\right]
$$

$$
\overset{R_3 \leftrightarrow R_3 - 2R_1}{\longrightarrow}
\left[\begin{array}{ccc|c}
1 & 1 & -3 & -18 \\
0 & 2 & -1 & 14 \\
0 & -3 & 9 & 60
\end{array}\right]
$$

$$
\overset{R_3 \leftrightarrow -\frac{1}{3}R_3}{\longrightarrow}
\left[\begin{array}{ccc|c}
1 & 1 & -3 & -18 \\
0 & 2 & -1 & 14 \\
0 & 1 & -3 & -20
\end{array}\right]
$$

$$
\overset{R_2 \leftrightarrow R_3}{\longrightarrow}
\left[\begin{array}{ccc|c}
1 & 1 & -3 & -18 \\
0 & 1 & -3 & -20 \\
0 & 2 & -1 & 14
\end{array}\right]
$$

$$
\overset{R_3 \leftrightarrow R_3 - 2R_2 \text{ and } R_1 \leftrightarrow R_1 - R_2}{\longrightarrow}
\left[\begin{array}{ccc|c}
1 & 0 & 0 & 2 \\
0 & 1 & -3 & -20 \\
0 & 0 & 5 & 54
\end{array}\right]
$$

$$
\overset{R_3 \leftrightarrow \frac{1}{5}R_3}{\longrightarrow}
\left[\begin{array}{ccc|c}
1 & 0 & 0 & 2 \\
0 & 1 & -3 & -20 \\
0 & 0 & 1 & \frac{54}{5}
\end{array}\right]
$$

$$
\overset{R_2 \leftrightarrow R_2 + 3R_3}{\longrightarrow}
\left[\begin{array}{ccc|c}
1 & 0 & 0 & 2 \\
0 & 1 & 0 & \frac{62}{5} \\
0 & 0 & 1 & \frac{54}{5}
\end{array}\right]
$$

Therefore, the solution is $(2, \frac{62}{5}, \frac{54}{5})$.

- **Example:** Use Gauss-Jordan reduction to find the solution to the following systems

1. $\begin{aligned} x &+ 2y &&= 0 \\ x &+ y &&= 6 \\ 3x &- 2y &&= 8 \end{aligned}$

   Ans: No solution.

2. $\begin{aligned} 2x_1 &- x_2 &+ 3x_3 &&= 24 \\ & 2x_2 &- x_3 &&= 14 \\ 7x_1 &- 5x_2 && &&= 6 \end{aligned}$

   Ans: (8,10,6)

3. $\begin{aligned} x &+ 2y &+ z &&= 8 \\ -3x &- 6y &- 3z &&= -21 \end{aligned}$

   Ans: No solution

4. $\begin{aligned} 4x &+ 12y &- 7z &- 20w &&= 22 \\ 3x &+ 9y &- 5z &- 28w &&= 30 \end{aligned}$

   Ans: $(100 - 3s + 96t, s, 54 + 52t, t)$

- **Homogeneous System:** A *homogeneous system* of equations is one in which the right hand side is 0. Example:

$$\begin{aligned} 2x_1 &+ 4x_2 &- 7x_3 &&= 0 \\ x_1 &- 3x_2 &+ 9x_3 &&= 0 \\ 6x_1 && &+ 9x_3 &&= 0 \end{aligned}$$

EVERY homogeneous system of linear equations is *consistent*, because the trivial solution (0,0,...,0) is always a solution. However, the system may have a unique solution (only the trivial solution) or infinitely many solutions. If there are fewer equations than variables, then the system will always have infinitely many solutions (there may be infinitely many solutions in other situations as well). Solving the above system, we have

$$\left[\begin{array}{ccc|c} 2 & 4 & -7 & 0 \\ 1 & -3 & 9 & 0 \\ 6 & 0 & 9 & 0 \end{array}\right] \quad \xrightarrow{R_1 \leftrightarrow R_2} \quad \left[\begin{array}{ccc|c} 1 & -3 & 9 & 0 \\ 2 & 4 & -7 & 0 \\ 6 & 0 & 9 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_2 - 2R_1 \text{ and } R_3 \leftrightarrow R_3 - 6R_1} \quad \left[\begin{array}{ccc|c} 1 & -3 & 9 & 0 \\ 0 & 10 & -25 & 0 \\ 0 & 18 & -45 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow \frac{1}{10} R_2} \quad \left[\begin{array}{ccc|c} 1 & -3 & 9 & 0 \\ 0 & 1 & -\frac{25}{10} & 0 \\ 0 & 18 & -45 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 \leftrightarrow R_1 + 3R_2 \text{ and } R_3 \leftrightarrow R_3 - 18R_2} \quad \left[\begin{array}{ccc|c} 1 & 0 & \frac{3}{2} & 0 \\ 0 & 1 & -\frac{5}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Hence, there is an infinite number of solutions with a free variable $x_3 = t$ and $x_1 = -\frac{3}{2}t$ and $x_2 = \frac{5}{2}t$. The solution set is then given by $(-\frac{3}{2}t, \frac{5}{2}t, t)$.

- **Problem:** Find values of a, b, and c (if possible) such that the system of linear equations has

  (a) a unique solution

  (b) no solution

  (c) an infinite number of solutions

  1.
  $$
  \begin{array}{rcrcrcr}
  x & + & y & & & = & 2 \\
  & & y & + & z & = & 2 \\
  x & & & + & z & = & 2 \\
  ax & + & by & + & cz & = & 0
  \end{array}
  $$

  2.
  $$
  \begin{array}{rcrcrcr}
  x & + & y & & & = & 0 \\
  & & y & + & z & = & 0 \\
  x & & & + & z & = & 0 \\
  ax & + & by & + & cz & = & 0
  \end{array}
  $$

- **Span:** Let $\mathbf{v}_a, \mathbf{v}_2, ..., \mathbf{v}_k$ be vectors in $\Re^n$. The *span* of these vectors is the set of all linear combinations of them and is denoted by $sp(\mathbf{v}_a, \mathbf{v}_2, ..., \mathbf{v}_k)$. In other words, a vector $\mathbf{x}$ is in the span of $\mathbf{v}_a, \mathbf{v}_2, ..., \mathbf{v}_k$ if there exists $c_1, c_2, ..., c_k$ such that

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_k\mathbf{v}_k.$$

- **Example:** Determine whether $\mathbf{b} = [1, -7, -4]$ is in the span of the vectors $\mathbf{v} = [2, 1, 1]$ and $\mathbf{w} = [1, 3, 2]$. To determine if $\mathbf{b}$ is in the span of $\{\mathbf{v}, \mathbf{w}\}$, we need to see if there exists scalars x_1 and x_2 such that

$$\mathbf{b} = x_1\mathbf{v} + x_2\mathbf{w}.$$

Written out in a rearranged order, this is equivalent to finding x_1 and x_2 such that

$$
x_1 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -7 \\ -4 \end{bmatrix}
$$

or there is a solution to the system

$$
\begin{bmatrix} 2 & 1 \\ 1 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -7 \\ -4 \end{bmatrix}
$$

The augmented system is

$$
\left[\begin{array}{cc|c} 2 & 1 & 1 \\ 1 & 3 & -7 \\ 1 & 2 & -4 \end{array} \right]
$$

which reduces to

$$
\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{array} \right]
$$

Hence, there is a solution, $x_1 = 2$, $x_2 = -3$ to the system so $\mathbf{b}$ is in $sp(\mathbf{v}, \mathbf{w})$.

- Let A be an mxn matrix. The linear system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if the vector $\mathbf{b}$ in $\Re^m$ is in the span of the columns of A.