Section 2.7
The Chain Rule
MATH 1190

- We know
$$\frac{d}{dx}(x^8) = 8x^7$$
But what about
$$y = (x^2+1)^8?$$

- We will use the **chain rule**. This first example will be done with u-substitution, and then from here on out, we will not use u-substitution exactly. Let u equal the *inside* function:
$$u = x^2 + 1$$
Then
$$y = (x^2+1)^8 = u^8$$
We have
$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$
So, for our example:

$\frac{dy}{dx}$	$=$	$\frac{dy}{du} \cdot \frac{du}{dx}$	from above
	$=$	$8u^7 \cdot (2x)$	because $\frac{d}{du}(u^8) = 8u^7$ and $\frac{d}{dx}(x^2+1) = 2x$
	$=$	$8(x^2+1)^7(2x)$	by substituting back in $u = x^2 + 1$.
	$=$	$16x(x^2+1)^7$	by simplifying

Note, since the function started in x, the derivative must have only x's in the answer.

- **Chain Rule:**
If y is a function of u and u is a function of x, then y is a function of x and
$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

- **Example:** Let $y = (2x)^7$. Notice, you have something *complicated* raised to a power, so you must use the chain rule. Think about taking the derivative of the outside function (leaving the inside function alone), and then multiplying by the derivative of the inside function:

$\frac{dy}{dx}$	$=$	$7(2x)^6 \cdot \frac{d}{dx}(2x)$	You bring the 7 in front, leaving the inside alone, and subtract 1 from the exponent - this gives $7(2x)^6$. Then you have to multiply by the derivative of the inside, because the inside is *complicated* (something other than just an x)
	$=$	$7(2x)^6 \cdot 2$	By taking the derivative of $2x$
	$=$	$14(2x)^6$	by simplifying

- **Example:** Let $f(x) = 3 + \sqrt{1-2x^2}$. Notice, on the second term you have something *complicated* under the square root (i.e. something complicated raised to the power 1/2, so you must use the chain rule on that portion. Think about taking the derivative of the outside function (leaving the inside function alone), and then multiplying by the derivative of the inside function:

$f'(x) =$	$\frac{d}{dx}(3) + \frac{d}{dx}\left((1-2x^2)^{1/2}\right)$	taking the derivative of each individual part
$=$	$0 + \frac{1}{2}(1-2x^2)^{-1/2}\frac{d}{dx}(1-2x^2)$	b/c $\frac{d}{dx}(3) = 0$ and on the second term, you bring the 1/2 in front, leave the inside alone and subtract 1 from the exponent. But, since the inside is *complicated*, you have to multiply by the derivative of the inside function
$=$	$\frac{1}{2}(1-2x^2)^{-1/2}(-4x)$	by taking the derivative of the inside function
$=$	$-2x(1-2x^2)^{-1/2}$	by simplifying
$=$	$\frac{-2x}{(1-2x^2)^{1/2}}$	to write with a positive exponent

- **Example:** Let $y = \sin(x^2e^x)$. Notice, you do not just have $\sin x$, you have sin (something complicated), so you have to use the chain rule:

$y' =$	$\cos(x^2e^x)\frac{d}{dx}(x^2e^x)$	the derivative of $\sin x$ is $\cos x$, so to take the derivative of $\sin(x^2e^x)$, you will have cos (what's inside) $\cdot$ derivative of what's inside
$=$	$\cos(x^2e^x)\left[x^2\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x^2)\right]$	using the product rule
$=$	$\cos(x^2e^x)\left[x^2e^x + e^x(2x)\right]$	taking the derivatives
$=$	$\cos(x^2e^x)\left[x^2e^x + 2xe^x\right]$	simplifying

- **Example:** Let $f(x) = (1+2x)e^{-2x}$. Notice you have two functions multiplied together, so you have to use the product rule. However, within the product rule, you need to notice that you have *e raised to something complicated*, so you have to use the chain rule as well.

$f'(x) =$	$(1+2x)\frac{d}{dx}(e^{-2x}) + e^{-2x}\frac{d}{dx}(1+2x)$	using the product rule
$=$	$(1+2x)e^{-2x}\frac{d}{dx}(-2x) + e^{-2x}(2)$	using the chain rule; $\frac{d}{dx}e^x = e^x$ so $\frac{d}{dx}e^{(\text{something } \textit{complicated})} = e^{(\text{something } \textit{complicated})}$.
$=$	$(1+2x)e^{-2x}(-2) + e^{-2x}(2)$	finishing taking derivative
$=$	$\frac{-2(1+2x)}{e^{2x}} + \frac{2}{e^{2x}}$	simplifying and no negative exponents
$=$	$\frac{-2-4x+2}{e^{2x}}$	combining as 1 fraction
$=$	$\frac{-4x}{e^{2x}}$	simplifying

- **Group Work:** Work on these problems in teams of 3-4. Find the derivative of the following functions.
  - Problems
    1. $f(x) = \sqrt{1-x}$
    2. $y = x^3(3x+4)^6$
    3. $y = (3x+1)^3(1+2x)^2$
    4. $f(x) = \frac{(x-1)^5}{x+2}$
    5. $f(x) = \frac{1}{(x^2+5)^3}$
    6. $y = \left(e^{\sin(t/2)}\right)^3$
    7. $f(x) = \sqrt{\frac{1-2x}{1+x}}$
  - Answers
    1. $f'(x) = \frac{-1}{2\sqrt{1-x}}$
    2. $y' = 18x^3(3x+4)^5 + 3x^2(3x+4)^6$
    3. $y' = 4(1+2x)(3x+1)^3 + 9(1+2x)^2(3x+1)^2$
    4. $f'(x) = \frac{5(x+2)(x-1)^4-(x-1)^5}{(x+2)^2}$ or $f'(x) = \frac{(x-1)^4(4x+11)}{(x+2)^2}$
    5. $f'(x) = \frac{-6x}{(x^2+5)^4}$
    6. $y' = \frac{3}{2}e^{3\sin(t/2)}\cos(t/2)$
    7. $f'(x) = \frac{-3}{2(1-2x)^{1/2}(1+x)^{3/2}}$