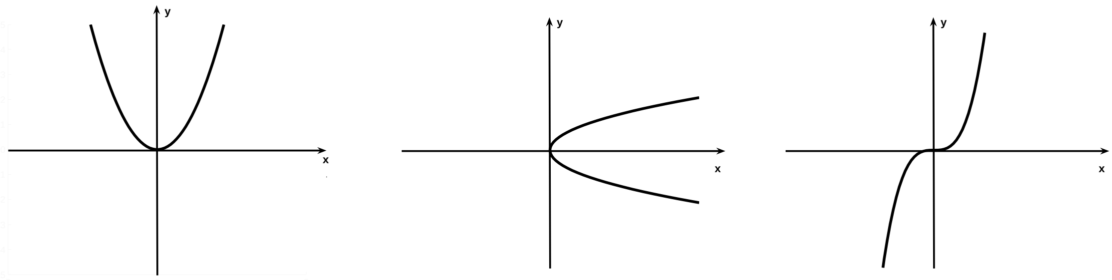
Section 2.9
Inverse Functions and Their Derivatives
MATH 1190

- Look at the following graphs:



Recall from college algebra that to be a *function*, the graph must pass the *vertical line test*, i.e., there cannot be two y values for a single x value. The vertical line test says that no vertical line can cross the graph more than once. If you look at the graphs, notice the one in the middle is NOT a function. However, the other two graphs represents graphs of function.

In order, for a graph to be a *one-to-one function*, i.e, for every x value there is one and only one y value, the graph must also pass the *horizontal line test.* The horizontal line test says that no horizontal line can cross the graph more than once. Notice, that even though the first graph is a function (passes the vertical line test), it is NOT a one-to-one function, because it does not pass the horizontal line test.

The only graph which represents a one-to-one function is the third graph. It passes both the vertical (which means it is a function) and horizontal (which means it is one-to-one) line tests.

- A one-to-one function f has an **inverse function**, denoted f^{-1}. Note that $f^{-1} \neq \frac{1}{f}$!!.
- Some facts about inverse functions:
  - the graph of an inverse function f^{-1} is found by reflecting the graph of f across the line $y = x$.
  - In order to find f^{-1} if given f, you must following the following steps: use the example

$$f(x) = \frac{1}{2}x + 1$$

1. Replace $f(x)$ with y:

$$y = \frac{1}{2}x + 1$$

2. Switch the x and y values (this denotes the reflection of the graph across the line $y = x$):

$$x = \frac{1}{2}y + 1$$

3. Solve for y:

$$x = \tfrac{1}{2}y + 1$$

$$x - 1 = \tfrac{1}{2}y$$

$$2(x - 1) = y$$

$$y = 2x - 2$$

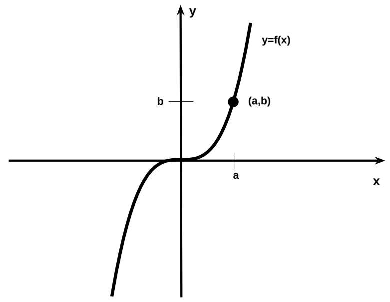
4. Replace y with $f^{-1}(x)$:

$$f^{-1}(x) = 2x - 2$$

  - Also recall that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.

- Notice that since we are switching x and y, we are switching the domains and ranges of the functions as well, so we have
  * domain of f^{-1} = range of f
  * range of f^{-1} = domain of f

- Notice, if $y = f(x)$, then $x = f^{-1}(y)$ since the values of x and y are switched. Therefore, if $f(a) = b$ like in the graph below, we also know $f^{-1}(b) = a$.



- Recall the example above: If

$$f(x) = \frac{1}{2}x + 1$$

then

$$f^{-1}(x) = 2x - 2.$$

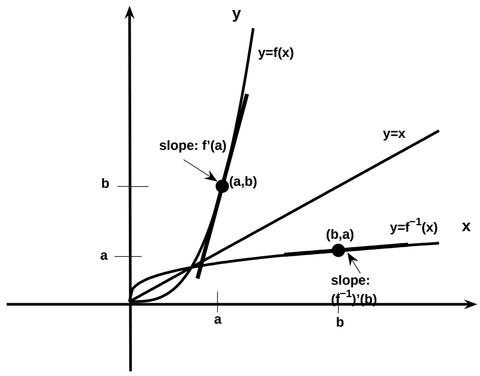
Thus,

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}(\frac{1}{2}x + 1) = \frac{1}{2}$$

and

$$\frac{d}{dx}(f^{-1}(x)) = \frac{d}{dx}(2x - 2) = 2$$

Thus, the derivatives are reciprocals of each other.



Notice in the graph above that the tangent lines to the graphs are reflected across the line $y = x$, thus we have:

$$\boxed{(f^{-1})'(b) = \frac{1}{f'(a)} = \frac{1}{f'(f^{-1}(b))}}$$

because $a = f^{-1}(b)$. In other words:

$$\boxed{\left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\frac{df}{dx}\right|_{x=a}}}$$

- **Example:** $f(x) = \frac{1}{5}x + 7$. Evaluate $\frac{df}{dx}$ at $x = -1$ and $\frac{df^{-1}}{dx}$ at $x = f(-1)$.
  Start by taking the derivative of the original function.
  $$\frac{df}{dx} = \frac{1}{5}$$
  so
  $$\frac{df}{dx}(-1) = \frac{1}{5}$$
  To find $\frac{df^{-1}}{dx}$ at $f(-1)$, we need to simply use the formula above:
  $$\left.\frac{df^{-1}}{dx}\right|_{x=b=f(a)} = \frac{1}{\left.\frac{df}{dx}\right|_{x=a}}$$
  In our problem this means:
  $$\left.\frac{df^{-1}}{dx}\right|_{x=f(-1)} = \frac{1}{\left.\frac{df}{dx}\right|_{x=-1}} = \frac{1}{\frac{1}{5}} = 5.$$
  In other words, we found $\left.\frac{df}{dx}\right|_{x=-1} = \frac{df}{dx}(-1) = \frac{1}{5}$, so the derivative of the inverse function at $x = f(-1)$ is simply the reciprocal. Note: there was no need to actually find the formula for $f^{-1}(x)$.

- **Example:** $f(x) = 2x^2$, $x \geq 0$. Find $\frac{df^{-1}}{dx}$ at $f(5)$.
  $$f'(x) = 4x$$
  So,
  $$f'(5) = 20$$
  Thus,
  $$\left.\frac{df^{-1}}{dx}\right|_{x=f(5)} = \frac{1}{20}$$