Section 2.3
Differentiation Rules
MATH 1190

- **Derivative of a Constant Function** Note that if you have a constant function, for example, $f(x) = 6$. The graph is a horizontal line. The slope of a horizontal line is 0, and the tangent to any horizontal line is also a horizontal line. Therefore, the slope of the tangent line (i.e., the derivative) at any point is 0, so you have

$$\boxed{\tfrac{d}{dx}(k) = 0 \text{ for any constant } k}$$

- **Power Rule** Using the definition of derivative, we can show:

$$\frac{d}{dx}(x^2) = 2x \quad \frac{d}{dx}(x^3) = 3x^2 \quad \frac{d}{dx}\frac{1}{x} = \frac{d}{dx}x^{-1} = -x^{-2} = -\frac{1}{x^2}$$

The pattern shows the following rule:

$$\boxed{\tfrac{d}{dx}(x^n) = nx^{n-1} \text{ where } n \text{ is any real number}}$$

- Examples: Find $f'(x)$ for the following functions.

1. $f(x) = x^5$
$$f'(x) = 5x^4$$
2. $f(x) = \sqrt{x}$ First rewrite $f(x) = x^{1/2}$. Then
$$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2x^{1/2}}$$
3. $f(x) = \sqrt[3]{x^2}$ First rewrite $f(x) = x^{2/3}$. Then
$$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3x^{1/3}}$$
4. $f(x) = \frac{1}{x^2}$ First rewrite $f(x) = x^{-2}$. Then
$$f'(x) = -2x^{-3} = \frac{-2}{x^3}$$

- **Constant Multiple Rule:** Since the constant does not depend on x, you can take it out:

$$\boxed{\tfrac{d}{dx}(c \cdot f(x)) = c \cdot \tfrac{d}{dx}f(x)}$$

Example: Find $f'(x)$ where
$$f(x) = \frac{x^5}{3}.$$
Notice that
$$f(x) = \frac{1}{3}x^5$$
so
$$f'(x) = \frac{1}{3}\frac{d}{dx}(x^5) = \frac{1}{3}(5x^4) = \frac{5}{3}x^4$$

$$\boxed{\frac{d}{dx}(cx^n) = n \cdot cx^{n-1}}$$

Example: Given

$$f(x) = \frac{2}{3}\sqrt[5]{x}$$

find $f'(x)$.

Rewrite $f(x) = \frac{2}{3}x^{1/5}$. Then

$$f'(x) = \frac{2}{5} \cdot \frac{1}{5}x^{-4/5} = \frac{2}{15x^{4/5}}$$

$$\boxed{\frac{d}{dx}(cx) = c}$$

Example: $\frac{d}{dx}(100x) = 100$

- **Sum and Difference Rule:**

$$\boxed{\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)}$$

- Examples: Find $f'(x)$.

  1. $f(x) = x^3 + 6x - 3$

  $$\frac{d}{dx}(x^3 + 6x - 3) = \frac{d}{dx}(x^3) + \frac{d}{dx}(6x) - \frac{d}{dx}(3) = 3x^2 + 6 - 0 = 3x^2 + 6$$

  2. $f(x) = 3x^4 + 2\sqrt{x} - \frac{2}{x^2}$ Rewrite:

  $$f(x) = 3x^4 + 2x^{1/2} - 2x^{-2}$$

  Then

  $$f'(x) = 12x^3 + 2 \cdot \frac{1}{2}x^{-1/2} - 2 \cdot (-2)x = 12x^3 + \frac{1}{x^{1/2}} + 4x$$

- **Higher Order Derivatives**

  - Second derivative: $f''(x) = \frac{d}{dx}(f'(x))$
  - Third derivative: $f'''(x) = \frac{d}{dx}(f''(x))$
  - Fourth derivative: $f^{(4)}(x) = \frac{d}{dx}(f'''(x))$
  - n^{th} derivative: $f^{(n)}(x) = \frac{d}{dx}(f^{(n-1)}(x)$
  - Example: $f(x) = \frac{x^3}{3} + \frac{x^2}{2} + \frac{x}{4} = \frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{4}x$
    Then the first derivative is found by

    $$f'(x) = x^2 + x + \frac{1}{4}$$

    The second derivative is

    $$f''(x) = 2x + 1$$

    Third derivative:

    $$f'''(x) = 2$$

    and the fourth derivative and all those greater than 4 are:

    $$f^{(n)}(x) = 0, \ n \geq 4$$

- Example: Find the equation for horizontal tangents to the curve

$$y = x^3 - 3x - 2$$

For a horizontal tangent line, the slope has to be 0. The slope of the tangent line is given by the derivative at a point c, i.e., $f'(c)$. First find the derivative and then find the values $x = c$ where $f'(c) = 0$.

$$f'(x) = 3x^2 - 3$$

We need to find c such that

$$f'(c) = 3c^2 - 3 = 0$$

Solving, we get

$$3(c^2 - 1) = 3(c - 1)(c + 1) = 0$$

Solving for c, we get $c = 1$, $c = -1$. We need to find the equation for the tangent line for each of these.

  - For $x = 1$. To find the y value, plug $x = 1$ into the original equation: $y = 1^3 - 3(1) - 2 = -4$. So, the equation for the tangent line is:

$$y - (-4) = 0(x - 1)$$

or

$$y = -4$$

  - For $x = -1$. To find the y value, plug $x = -1$ into the original equation: $y = (-1)^3 - 3(-1) - 2 = 0$. So, the equation for the tangent line is:

$$y - 0 = 0(x - 1)$$

or

$$y = 0$$

- Example: Find $\frac{ds}{dt}$ if

$$s = \frac{t^2 + 5t - 1}{t^2}$$

We only know what to do if we have functions adding or subtracting each other, so we must split it up

$$s = \frac{t^2}{t^2} + \frac{5t}{t^2} - \frac{1}{t^2} = 1 + \frac{5}{t} - \frac{1}{t^2} = 1 + 5t^{-1} - t^{-2}$$

Then

$$\frac{ds}{dt} = -5t^{-2} + 2t^{-3} = -\frac{5}{t^2} + \frac{2}{t^3}$$

- **Group Work - worksheet Part II**

- **Derivatives of Products**: If $u(x)$ and $v(x)$ are differentiable at x, then

$$\boxed{\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}}$$

The way I look at it

$$\frac{d}{dx}(\text{first} \cdot \text{second}) = \text{first}\frac{d}{dx}(\text{second}) + \text{second}\frac{d}{dx}(\text{first})$$

- Example: Find y' if

$$y = (3 + x^2)x^4$$

Using the product rule, you have

$$\begin{aligned} \frac{dy}{dx} &= (3+x^2)\frac{d}{dx}(x^4) + x^4\frac{d}{dx}(3+x^2) \\ &= (3+x^2)(4x^3) + x^4(2x) \\ &= 12x^3 + 6x^5 \end{aligned}$$

- Example: Find y' if

$$y = \left(x + \frac{1}{x}\right)\left(x - \frac{1}{x} + 1\right)$$

We can rewrite

$$y = (x + x^{-1})(x - x^{-1} + 1)$$

then

$$\begin{aligned} y' &= (x+x^{-1})\frac{d}{dx}(x - x^{-1} + 1) + (x - x^{-1} + 1)\frac{d}{dx}(x + x^{-1}) \\ &= (x+x^{-1})(1 + x^{-2}) + (x - x^{-1} + 1)(1 - x^{-2}) \\ &= \left(x + \frac{1}{x}\right)\left(1 + \frac{1}{x^2}\right) + \left(x - \frac{1}{x} + 1\right)\left(1 - \frac{1}{x^2}\right) \end{aligned}$$

- Suppose u and v are differentiable functions of x and $u(1) = 2$, $u'(1) = 0$, $v(1) = 5$, and $v'(1) = -1$. Find

$$\frac{d}{dx}(7v(x) - 2u(x))$$

and

$$\frac{d}{dx}(u(x) \cdot v(x))$$

at $x = 1$.

$$\frac{d}{dx}(7v(x) - 2u(x)) = 7v'(x) - 2u'(x)$$

so at $x = 1$, we had

$$7v'(1) - 2u'(1) = 7(-1) - 2(0) = -7$$

For the second one:

$$\frac{d}{dx}(u(x) \cdot v(x)) = u(x)v'(x) + v(x)u'(x)$$

so at $x = 1$, we have

$$u(1)v'(1) + v(1)u'(1) = 2(-1) + 5(0) = -2$$

- **Quotient Rule**: If $u(x)$ and $v(x)$ are differentiable at x, then

$$\boxed{\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}}$$

The way I look at it

$$\frac{d}{dx}\left(\frac{\text{top}}{\text{bottom}}\right) = \frac{\text{bottom}\frac{d}{dx}(\text{top}) - \text{top}\frac{d}{dx}(\text{bottom})}{(\text{bottom})^2}$$

- Example: Find $f'(x)$ if

$$f(x) = \frac{x^2}{2x-1}$$

Using the quotient rule:

$$\begin{aligned} f'(x) &= \frac{(2x-1)\frac{d}{dx}(x^2) - x^2\frac{d}{dx}(2x-1)}{(2x-1)^2} \\ &= \frac{(2x-1)(2x) - x^2(2)}{(2x-1)^2} \\ &= \frac{4x^2-2x-2x^2}{(2x-1)^2} \\ &= \frac{2x^2-2}{(2x-1)^2} \end{aligned}$$

- Example: Find $s'(t)$ if

$$s(t) = \frac{t^2+t+1}{t+1}$$

Using the quotient rule,

$$\begin{aligned} s'(t) &= \frac{(t+1)\frac{d}{dt}(t^2+t+1) - (t^2+t+1)\frac{d}{dt}(t+1)}{(t+1)^2} \\ &= \frac{(t+1)(2t+1) - (t^2+t+1)(1)}{(t+1)^2} \\ &= \frac{2t^2+3t+1-(t^2+t+1)}{(t+1)^2} \\ &= \frac{2t^2+3t+1-t^2-t-1)}{(t+1)^2} \\ &= \frac{t^2+2t}{(t+1)^2} \end{aligned}$$

- Example: If you have

$$f(x) = \frac{5x+1}{2\sqrt{x}}$$

you can use the quotient rule, but the easiest way to find $f'(x)$ is to simplify the fraction.

$$f(x) = \frac{5x}{2x^{1/2}} + \frac{1}{2x^{1/2}} = \frac{5}{2}x^{1/2} + \frac{1}{2}x^{-1/2}$$

So,

$$f'(x) = \frac{5}{4}x^{-1/2} - \frac{1}{4}x^{-3/2} = \frac{5}{4x^{1/2}} - \frac{1}{4x^{3/2}}$$

- **Group Work - worksheet Part III**