Section 2.6
Exponential Functions
MATH 1190

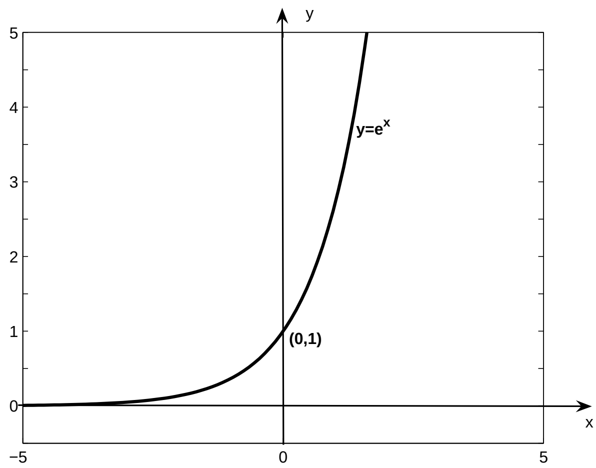
- Exponential functions have the form $f(x) = a^x$, where a is the base, x is the exponent; $a \neq 1$, $a > 0$.

$$a^n = \underbrace{a \cdot a \cdot a \cdots a}_{n \text{ times}}$$

- The *natural* exponential function is given by $f(x) = e^x$ where

$$e^1 \approx 2.718281828...$$

Graph of $f(x) = e^x$:



- Rules for Exponents: If $a > 0$ and $b > 0$, the the following rules hold for all real numbers x and y.

  1. $a^x \cdot a^y = a^{x+y}$
  2. $\frac{a^x}{a^y} = a^{x-y}$
  3. $(a^x)^y = a^{xy}$
  4. $a^x \cdot b^x = (ab)^x$
  5. $\frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x$

- Derivative of the *natural* exponential function:

$$\boxed{\frac{d}{dx} e^x = e^x}$$

- **Example:** Let $y = xe^{-x}$, find y'.

$$y = xe^{-x} = x \cdot \frac{1}{e^x} = \frac{x}{e^x}$$

$$
\begin{aligned}
y' &= \tfrac{e^x \frac{d}{dx}(x) - x\frac{d}{dx}(e^x)}{(e^x)^2} && \text{Quotient Rule} \\
&= \tfrac{e^x(1) - xe^x}{e^{2x}} && \text{Taking the derivative of the individual pieces} \\
&= \tfrac{e^x - xe^x}{e^{2x}} && \text{Simplifying} \\
&= \tfrac{e^x(1-x)}{e^{2x}} && \text{Factoring out } e^x \\
&= \tfrac{1-x}{e^x} && \text{Simplifying}
\end{aligned}
$$

- **Example:** Let $f(x) = \sqrt[3]{x^{9.6}} + 2e^{1.3}$, find $f'(x)$.

$$f(x) = (x^{9.6})^{1/3} + 2e^{1.3} = x^{9.6/3} + 2e^{1.3} = x^{3.2} + 2e^{1.3}$$

$$f'(x) = 3.2x^{3.2-1} + 0 = 3.2x^{2.2}$$

because $2e^{1.3}$ does not have an x in it, it is a constant and $\frac{d}{dx}(c) = 0$ for any constant c. Also, $\frac{d}{dx}(x^{3.2})$ is taken like any other x^n: $\frac{d}{dx}x^n = nx^{n-1}$.

- **Group Work:**

1. In groups, find the derivative of the following functions:

   (a) $y = \frac{1}{\sqrt[5]{x^3}} + \pi^{3/2} + e^x$

   Answer: $y' = \frac{-3}{5x^{8/5}} + e^x$

   (b) $r = e^\theta \left(\frac{1}{\theta^2} + \theta^{-\pi/2}\right)$

   Answer: $r' = e^\theta \left(\frac{-2}{\theta^3} - \frac{\pi}{2\theta^{\pi/2+1}}\right) + e^\theta \left(\frac{1}{\theta^2} + \frac{1}{\theta^{\pi/2}}\right)$

2. Find the first and second derivatives of the following function

$$y = \frac{1 - e^{-x}}{x^2 + 1}$$

Answer:
$y' = \frac{-2xe^x + 2x + x^2 + 1}{e^x(x^2+1)^2}$

$y'' = \frac{e^x(x^2+1)^2[-2xe^x - 2e^x + 2 + 2x] - (-2xe^x + 2x + x^2 + 1)(4xe^x(x^2+1) + e^x(x^2+1)^2)}{e^{2x}(x^2+1)^4}$