

Section 4.5

Fundamental Theorem of Calculus

MATH 1190

- We will start with the **Fundamental Theorem of Calculus, Part II**:

$$\int_a^b f(x) \, dx = F(b) - F(a)$$

where F is an antiderivative of f .

- **Example:**

$$\begin{aligned} \int_1^2 3x^2 \, dx &= 3 \int_1^2 x^2 \, dx \\ &= 3 \left. \frac{x^3}{3} \right|_1^2 \\ &= x^3 \Big|_1^2 \\ &= 2^3 - 1^3 \\ &= 7 \end{aligned}$$

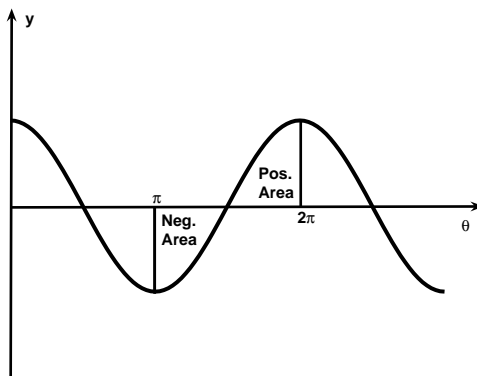
- **Example:**

$$\begin{aligned} \int_1^e \left(x - \frac{1}{x}\right) \, dx &= \left. \frac{x^2}{2} - \ln x \right|_1^e \\ &= \left(\frac{e^2}{2} - \ln e\right) - \left(\frac{1^2}{2} - \ln 1\right) \\ &= \frac{e^2}{2} - 1 - \frac{1}{2} \\ &= \frac{e^2}{2} - \frac{3}{2} \end{aligned}$$

- **Example:**

$$\begin{aligned} \int_{\pi}^{2\pi} \cos \theta \, d\theta &= \sin \theta \Big|_{\pi}^{2\pi} \\ &= \sin(2\pi) - \sin \pi \\ &= 0 \end{aligned}$$

- **Integral vs. total area:** Notice the example above. The integral of $\cos \theta$ as θ goes from π to 2π is 0. Let's relate this to area.



The reason why the integral is 0 is because there is as much area above the x -axis (considered positive) as below the x -axis (considered negative). Thus they cancel out and make the integral 0. However,

one may be interested in the **total area** between the x -axis and the curve over the interval $[a, b]$. To find total area, we must follow the following steps:

1. Subdivide the interval $[a, b]$ at the zeros (where the graph crosses the x -axis).
2. Integrate f over each subinterval.
3. Add the absolute values of the integrals.

- **Example** Find the total area under the graph of $y = \cos \theta$ on the interval $[\pi, 2\pi]$. Look at the graph above. We need to find the point(s) at which the graph crosses the x -axis. I.e., we need to find where $\cos \theta = 0$ on the interval $[\pi, 2\pi]$. The only angle which gives $\cos \theta = 0$ along this interval is $\theta = \frac{3\pi}{2}$. So, to find the total area look at:

$$\int_0^{3\pi/2} \cos \theta \, d\theta = \sin \theta \Big|_0^{3\pi/2} = \sin(3\pi/2) - \sin 0 = -1$$

and

$$\int_{3\pi/2}^{2\pi} \cos \theta \, d\theta = \sin \theta \Big|_{3\pi/2}^{2\pi} = \sin(2\pi) - \sin(3\pi/2) = 1$$

So, the total area is:

$$|-1| + 1 = 2$$

- **Group Work:**

– **Problems:** Evaluate the following

1. $\int_0^6 (x^2 - 8x + 17) \, dx$
2. $\int_1^8 (\sqrt[3]{x} - 2) \, dx$
3. $\int_0^4 (2v + 5)(3v - 1) \, dv$
4. $\int_1^9 \frac{3x-2}{\sqrt{x}} \, dx$
5. Find the total area under the graph of $y = 2 - x - x^2$ on the interval $[0, 4]$.

– **Answers:**

1. 30
2. $-\frac{11}{4}$
3. 212
4. 44
5. $\frac{71}{3} \approx 23.7$

- **Fundamental Theorem of Calculus, Part I:** If

$$g(x) = \int_a^x f(t) \, dt$$

then

$$g'(x) = f(x)$$

In other words, the derivative and integral cancel each other out; if we integrate a function and then differentiate it, we get the same function we started with.

- **Example:** If

$$g(x) = \int_0^x \sqrt{1+t^2} \, dt$$

find $g'(x)$.

$$g'(x) = \sqrt{1+x^2}$$

using the fundamental theorem of calculus.

- **Example:** If

$$g(x) = \int_x^{10} \tan \theta \, d\theta$$

find $g'(x)$.

Recall,

$$\int_a^b f(x) \, dx = - \int_b^a f(x) \, dx,$$

so,

$$g(x) = \int_x^{10} \tan \theta \, d\theta = - \int_{10}^x \tan \theta \, d\theta$$

Thus, using the fundamental theorem of calculus, we have

$$g'(x) = -\tan x$$

- **Example:** If

$$g(x) = \int_{\cos x}^{\sin x} (1 + v^2)^{10} \, dv$$

find $g'(x)$.

Recall,

$$\int_a^b f(x) \, dx + \int_b^c f(x) \, dx = \int_a^c f(x) \, dx$$

So, we can write

$$g(x) = \int_{\cos x}^0 (1 + v^2)^{10} \, dv + \int_0^{\sin x} (1 + v^2)^{10} \, dv = - \int_0^{\cos x} (1 + v^2)^{10} \, dv + \int_0^{\sin x} (1 + v^2)^{10} \, dv$$

Then using the chain rule, we have

$$\begin{aligned} g'(x) &= -(1 + (\cos x)^2)^{10} \frac{d}{dx}(\cos x) + (1 + (\sin x)^2)^{10} \frac{d}{dx}(\sin x) \\ &= -(1 + (\cos^2 x))^{10}(-\sin x) + (1 + (\sin^2 x))^{10}(\cos x) \\ &= (1 + (\cos^2 x))^{10}(\sin x) + (1 + (\sin^2 x))^{10}(\cos x) \end{aligned}$$

- **Group Work:**

– **Problems:** Find y' if

1. $y = \int_1^x \frac{1}{t} \, dt$
2. $y = \int_0^{x^2} \cos(\sqrt{t}) \, dt$
3. $y = \int_{2^x}^1 \sqrt[3]{t} \, dt$

– **Answers:**

1. $y' = \frac{1}{x}$
2. $y' = 2x \cos x$
3. $y = -2^x \sqrt[3]{2^x} \ln 2 = -2^{\frac{4}{3}x} \ln 2$