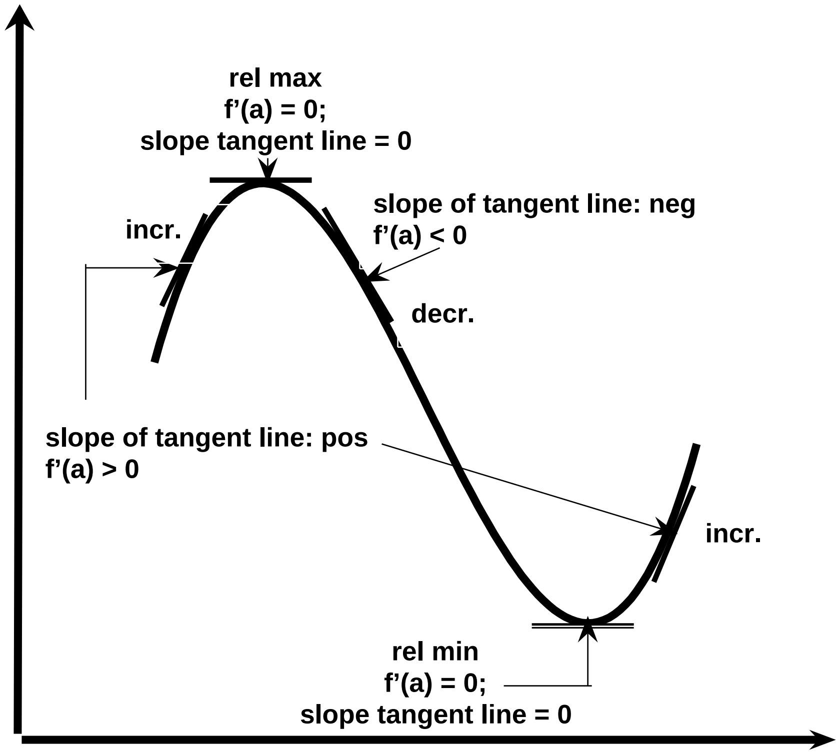
# Section 3.1(part), 3.3-3.4

## Critical Numbers, Extreme Values, Increasing/Decreasing, Concave Up/Down

**MATH 1190**




- **Increasing/Decreasing**
  - *Increasing:* If $f'(a) > 0$, then $f$ is increasing at $x = a$.
    If $f'(x) > 0$ for all $x$ on an interval, then $f$ is increasing on that interval.
  - *Decreasing:* If $f'(a) < 0$, then $f$ is decreasing at $x = a$.
    If $f'(x) < 0$ for all $x$ on an interval, then $f$ is decreasing on that interval.
- **Critical Number:** When $f'(c) = 0$ or $f'(c)$ is undefined for some point $x = c$, then $f$ is neither increasing nor decreasing and $x = c$ is called a *critical number of $f$*.
- **A function ONLY changes from increasing to decreasing at CRITICAL numbers.**
- **Example:** Let

$$f(x) = x^2 - 1$$

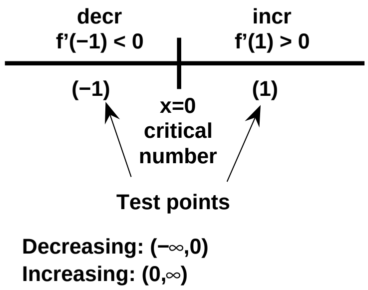
1. Find all critical numbers of $f$.

$$f'(x) = 2x$$

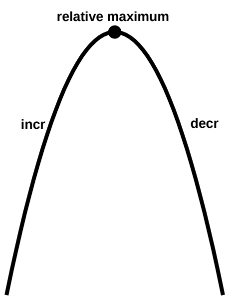
   Set $f'(x) = 0$ to find the critical numbers.

$$2x = 0 \rightarrow x = 0$$

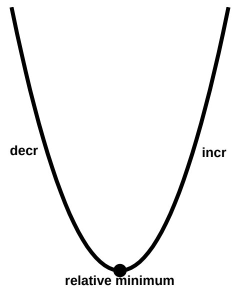
2. Find the intervals where $f$ is increasing and where $f$ is decreasing.
   - Put the critical number on a number line.
   - Pick test points in each interval.
   - Determine if the *derivative* at each test point is positive or negative:
     $f'(\text{test point}) > 0$ then it is increasing on the *entire* interval.
     $f'(\text{test point}) < 0$ then it is decreasing on the *entire* interval.
   - Write the interval where the function is increasing/decreasing in interval notation.




- **Relative Maximum/Minimum:** At a critical number $x = c$, you may have a
  - *Relative maximum:* At a relative maximum point, the $y$ value of the point is greater than any of the $y$ values immediately near it.

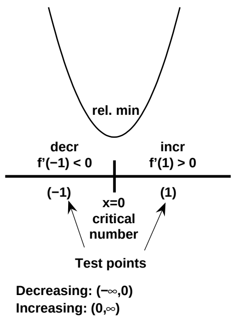


- *Relative minimum:* At a relative minimum point, the $y$ value of the point is less than any of the $y$ values immediately near it.

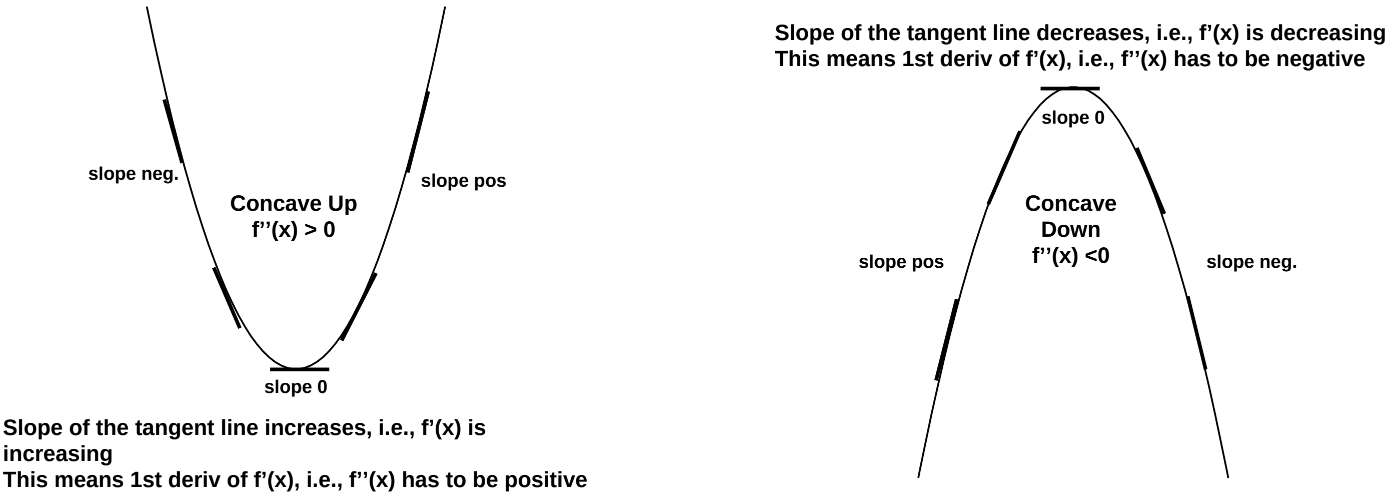



- **First Derivative Test for Relative Maximum or Minimum:** Let $x = c$ be a critical number of $f(x)$, then the point $(c, f(c))$ is a
  - *relative maximum* if $f'(x) > 0$ (incr.) to the left of $x = c$ and $f'(x) < 0$ (decr) to the right of $x = c$.
  - *relative minimum* if $f'(x) < 0$ (decr.) to the left of $x = c$ and $f'(x) > 0$ (incr) to the right of $x = c$.

- **Back to the example:** since $f$ is decr. to the left and incr. to the right of $x = 0$, then $(0, f(0)) = (0, -1)$ is a *relative minimum* of $f$.



- **Concavity:**




  - *Concave up:* A function $f$ is concave up at a point $x = a$ if $f''(a) > 0$.
  - *Concave down:* A function $f$ is concave down at a point $x = a$ if $f''(a) < 0$.

- **Point of inflection:** Any point at which the graph of a function changes concavity is called a *point of inflection.*

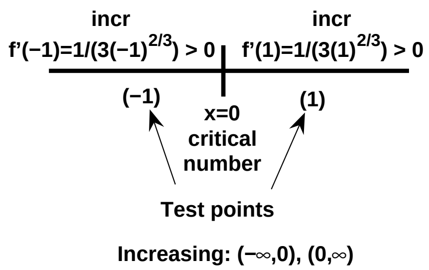
  **Possible points of inflection** occur where

  1. $f''(x) = 0$
  2. $f''(x)$ undefined.

- **Example:** Examine the function
$$f(x) = \sqrt[3]{x}$$
  Let's go through all the questions.

  - *critical numbers:* To find critical numbers, we want to see where $f'(x) = 0$ or $f'(x)$ is undefined.
$$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$$
    So, to figure out where a fraction is equal to 0, you would look at where the numerator is equal to 0. Here the numerator is 1, but $1 \neq 0$, so there is no place where $f'(x) = 0$. To figure out where $f'(x)$ is undefined, you look where the denominator is equal to 0:
$$\begin{aligned} 3x^{2/3} &= 0 \quad \text{the denominator of } f'(x) \text{ set equal to 0} \\ x^{2/3} &= 0 \quad \text{by dividing by 3} \\ (x^{2/3})^{3/2} &= 0^{3/2} \quad \text{by raising each side to the 3/2} \\ x &= 0 \quad \text{by simplifying} \end{aligned}$$
    So, the only critical number is $x = 0$.

– *intervals increasing and decreasing:* Pick test points in each interval and look at the value of $f'(x)$ in each interval.

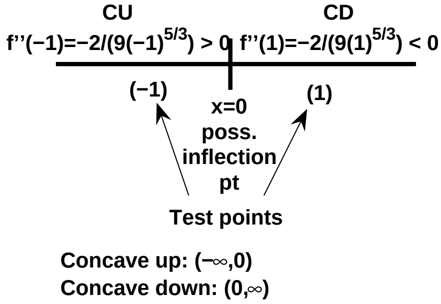



– *Relative Maximum or Minimum:* Since the function is increasing both to the left and right of the critical number, there is NO relative maximum or minimum.

– *Possible points of inflection:* To find possible points of inflection, find where $f''(x) = 0$ or $f''(x)$ is undefined.

$$f''(x) = \frac{-2}{9}x^{-5/3} = \frac{-2}{9x^{5/3}}$$

There is no $x$ which gives $f''(x) = 0$, because the is no $x$ which will make the numerator equal to 0. $x = 0$ makes $f''(x)$ undefined. So, $x = 0$ is the only possible inflection point. Do the same as you did with increasing and decreasing to figure out concavity, but with the *second* derivative!



  - *Sketch the graph* In class.

- **Second Derivative Test for Relative Maximum or Minimum:** Let $x = c$ be a critical number of $f(x)$, then the point $(c, f(c))$ is a
  - *relative maximum* if $f''(c) > 0$ (concave down at $x = c$).
  - *relative minimum* if $f''(c) < 0$ (concave up at $x = c$).

  The second derivative test FAILS if $f''(c) = 0$ or $f''(c)$ is undefined.

- **Group Work:**
  1. Find
     - The critical numbers of $f$.
     - The intervals where $f$ is increasing and decreasing.
     - Any relative maximum and minimum.
     - Any possible inflection points.
     - Intervals where $f$ is concave up and concave down.

     for each of the following:

     (a) $f(x) = x^3 - 27x$
     (b) $f(x) = 2x^3 - 3x^2 - 12x - 2$
     (c) $f(x) = x^4 + 2x^3$
  2. Sketch a graph of the function with the following properties.
     - $f'(-3) = 0$
     - $f''(-3) < 0$
     - $f(-3) = 8$
     - $f'(9) = 0$
     - $f''(9) > 0$
     - $f(9) = -6$
     - $f''(2) = 0$
     - $f(2) = 1$