Section 2.8
Implicit Differentiation
MATH 1190

- Previously, we dealt with functions where y was an ***explicit function*** of x (y was was written explicitly in terms of x). For example,

  1. $y = 3x^2 - 4$
  2. $y = (4x - 5)^{1/2}$

- But, y may be an ***implicit function*** of x (y not written explicitly in terms of x). For example,

  1. $3xy - 2 = 0$
  2. $y^3 - x^2 + xy = 3$

  Note that even with implicit functions of x, y still depends on x and the change in y depends on how x changes.

- In number 1 above, we could rewrite the equation, so y was written as an explicit function of x:

$$3xy - 2 = 0$$

$$3xy = 2$$

$$y = \tfrac{2}{3x}$$

  In this form, we can rewrite y as

$$y = \frac{2}{3}x^{-1}$$

  Hence, the derivative would be

$$y' = -\frac{2}{3}x^{-2} = \frac{-2}{3x^2}$$

- In number 2, we cannot rewrite y as an explicit function of x; however, we can still find $\frac{dy}{dx}$ using implicit differentiation.

- Implicit differentiation uses a *special form of the chain rule.* We assume y is a function of x (i.e. y = f(x)), so

$$\frac{d}{dx}y^n = \frac{d}{dx}(f(x))^n = n(f(x))^{n-1} \cdot f'(x) = ny^{n-1}\frac{dy}{dx}$$

  More simply,

$$\frac{d}{dx}y^n = ny^{n-1} \cdot \frac{dy}{dx}$$

- **Simple example:**

$$\frac{d}{dx}y^3 = 3y^2\frac{dy}{dx}$$

  Notice, we are taking the derivative with respect to x, so we take the derivative like we normally do and get $3y^2$, but since we have y and we are taking the derivative with respect to x, we have to multiply by $\frac{dy}{dx}$.

- **Another simple example:**

$\frac{d}{dx}(xy)$	$=$	$x\frac{d}{dx}y + y\frac{d}{dx}(x)$	using the product rule
	$=$	$x \cdot (1) \cdot \frac{dy}{dx} + y \cdot (1)$	b/c in the first term, we need to take the derivative of y with respect to x. The normal derivative would be 1 but since it is the derivative of y with respect to x we have to multiply by the term $\frac{dy}{dx}$. In the second term, we simply are taking the derivative of x with respect to x, so we simply get 1
	$=$	$x\frac{dy}{dx} + y$	simplifying

- Now, let's use implicit differentiation to differentiate the first equation above:

$$3xy - 2 = 0$$

We will break this into several steps:

- **Step 1:** Differentiate both sides (all terms) of the equation with respect to x:

$$\frac{d}{dx}(3xy) - \frac{d}{dx}(2) = \frac{d}{dx}(0)$$

$$3x\frac{d}{dx}(y) + y\frac{d}{dx}(3x) - 0 = 0 \qquad \text{using product rule}$$

$$3x\frac{dy}{dx} + y(3) = 0$$

$$3x\frac{dy}{dx} + 3y = 0$$

- **Step 2:** Collect all terms containing $\frac{dy}{dx}$ on one side and all other terms on the other side.

$$3x\frac{dy}{dx} = -3y$$

- **Step 3:** Factor out $\frac{dy}{dx}$ from all terms containing it. Note that this step is not necessary in this problem.
- **Step 4:** Solve for $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{-3y}{3x}$$

$$\frac{dy}{dx} = \frac{-y}{x}$$

Notice that this is equivalent to what we got above. Since we know y can be written explicitly (from above) as $y = \frac{2}{3x}$, then

$$\frac{dy}{dx} = \frac{-y}{x} = \frac{-\frac{2}{3x}}{x} = \frac{-2}{3x \cdot x} = \frac{-2}{3x^2}.$$

This is the same as above.

- However, in example 2, we cannot write out y explicitly, but we may still need to understand how y changes as a function of x (remember, $\frac{dy}{dx}$ means the change of y with respect to x). Therefore, in example 2, there is no other way to find $\frac{dy}{dx}$ except by using implicit differentiation. Let's go through the same steps with this problem:

$$y^3 - x^2 + xy = 3$$

- **Step 1:** Differentiate both sides (all terms) of the equation with respect to x:

$$\frac{d}{dx}(y^3 - x^2 + xy) = \frac{d}{dx}(3)$$

$$\frac{d}{dx}(y^3) - \frac{d}{dx}(x^2) + \underbrace{\frac{d}{dx}(xy)}_{\text{Product rule}} = \frac{d}{dx}(3)$$

$$3y^2\frac{dy}{dx} - 2x + x\frac{d}{dx}(y) + y\frac{d}{dx}(x) = 0$$

$$3y^2\frac{dy}{dx} - 2x + x\frac{dy}{dx} + y(1) = 0$$

$$3y^2\frac{dy}{dx} - 2x + x\frac{dy}{dx} + y = 0$$

- **Step 2:** Collect all terms containing $\frac{dy}{dx}$ on one side and all other terms on the other side.

$$3y^2\frac{dy}{dx} + x\frac{dy}{dx} = 2x - y$$

- **Step 3:** Factor out $\frac{dy}{dx}$ from all terms containing it.

$$\frac{dy}{dx}\left(3y^2+x\right)=2x-y$$

- **Step 4:** Solve for $\frac{dy}{dx}$

$$\frac{dy}{dx}=\frac{2x-y}{3y^2+x}$$

- **Example:** Obtain $\frac{dy}{dx}$ using implicit differentiation.

$$x^2-y^2=2x-5$$

Go through the steps:

$\frac{d}{dx}(x^2-y^2)=\frac{d}{dx}(2x-5)$

$\frac{d}{dx}(x^2)-\frac{d}{dx}(y^2)=\frac{d}{dx}(2x)-\frac{d}{dx}(5)$

$2x-2y\frac{dy}{dx}=2-0$ Notice that we have $\frac{dy}{dx}$ in the second term, because we took the derivative of a y term w.r.t x

$2x-2y\frac{dy}{dx}=2$ simplifying

So, you get

$$-2y\frac{dy}{dx}=2-2x$$

$$\frac{dy}{dx}=\frac{2-2x}{-2y}$$

$$\frac{dy}{dx}=\frac{-2(-1+x)}{-2y}$$

$$\frac{dy}{dx}=\frac{(-1+x)}{y}$$

$$\frac{dy}{dx}=\frac{x-1}{y}$$

- **Example:** Find the equation of the tangent line to the curve

$$y-3xy=22$$

at the point (4,-2).

In order to find the equation for the tangent line, we first need to find the slope of the tangent line. Remember,

$$m_{\tan}=\frac{dy}{dx}\text{ at }x=a$$

So, we first need to find the derivative $\frac{dy}{dx}$. To do this, use implicit differentiation:

$\frac{d}{dx}(y - 3xy) = \frac{d}{dx}(22)$

$\frac{d}{dx}(y) - \frac{d}{dx}(3xy) = 0$

$(1) \cdot \frac{dy}{dx} - (3x\frac{d}{dx}(y) + y\frac{d}{dx}(3x)) = 0$

$\frac{dy}{dx} - (3x\frac{dy}{dx} + y(3)) = 0$

$\frac{dy}{dx} - 3x\frac{dy}{dx} - 3y = 0$

$\frac{dy}{dx} - 3x\frac{dy}{dx} = 3y$

$\frac{dy}{dx}(1 - 3x) = 3y$

$\frac{dy}{dx} = \frac{3y}{1-3x}$

So, we have $\frac{dy}{dx} = \frac{3y}{1-3x}$. To find the slope, we need to plug in the point given: $x = 4$, $y = -2$ to get

$$\frac{dy}{dx} = \frac{3(-2)}{1-3(4)} = \frac{6}{11}$$

Then, the equation for the line is

$$y - y_1 = m_{\text{tan}}(x - x_1)$$

or

$$y - (-2) = \frac{6}{11}(x - 4)$$

Simplifying, we get

$$y = \frac{6}{11}x - \frac{2}{11}$$

as the equation for the tangent line.

- **Group Work:**
  - **Problems:**
    1. Find $\frac{dy}{dx}$ using implicit differentiation.
       (a) $y^5 + x^2y^3 = 1 + ye^{x^2}$
       (b) $1 + x = \sin(xy^2)$
       (c) $y\sin(x^2) = x\sin(y^2)$
    2. If $g(x) + x\sin(g(x)) = x^2$, find $g'(0)$.
    3. Use implicit differentiation to find the equation for the tangent line to

       $$x^2 + 2xy - y^2 + x = 2$$

       at the point (1,2).
  - **Answers:**
    1. (a) $\frac{dy}{dx} = \frac{2xye^{x^2} - 2xy^3}{5y^4 + 3x^2y^2 - e^{x^2}}$
       (b) $\frac{dy}{dx} = \frac{1 - y^2\cos(xy^2)}{2xy\cos(xy^2)}$
       (c) $\frac{dy}{dx} = \frac{\sin(y^2) - 2xy\cos(x^2)}{\sin(x^2) - 2xy\cos(y^2)}$
    2. $g'(0) = -\sin(g(0))$.
    3. $y = \frac{7}{2}x - \frac{3}{2}$