Section 4.6 and 4.7
Integration by Substitution
MATH 1190

- Suppose you have the integral

$$\int 2x(x^2+2)^3\,dx$$

Up to this point, we have only learned how to integrate functions which are adding or subtracting. It would be extremely time consuming to multiply all this out so we have only addition and subtraction of terms. In addition, there is room for ALOT of errors. However, this is something complicated raised to a power. When we had something complicated raised to a power and wanted to differentiate, we used the chain rule, so we can use the "reverse" of the chain rule here - the *substitution rule.*

- Let's examine how we use the substitution rule with the above example:

$$\int 2x(x^2+2)^3\,dx$$

1. Let u equal the complicated part (that is being raised to a power or inside of a trig function, etc.). So, in this case, we would set

$$u = x^2 + 2$$

2. Find $\frac{du}{dx}$.

$$\frac{du}{dx} = 2x$$

3. Solve for du:

$$du = 2x\,dx$$

4. Now, look back at the original problem and substitute in u where it belongs:

$$\int 2x(x^2+2)^3\,dx = \int 2xu^3\,dx$$

5. Our goal is to write everything in terms of u. Group all the non-u terms together

$$\int u^3\,2xdx$$

See how the non-u terms compare to du. Since $du = 2x\,dx$, you can substitute du in for these terms,

$$\int u^3\,du$$

6. Now, everything is in terms of u, so we can integrate this "simple" problem:

$$\int u^3\,du = \frac{u^4}{4} + C$$

7. Since the problem started with x, it must end wit everything in terms of x. So, substitute back in for u, $u = x^2 + 2$:

$$\int 2x(x^2+2)^3\,dx = \frac{(x^2+2)^4}{4} + C$$

8. Check your answer is the last step as always!

- **Example**

$$\int (3x^2-2)(x^3-2x)^4\ dx$$

Let u be the complicated part, i.e.,

$$u = x^3 - 2x$$

So,

$$\frac{du}{dx} = 3x^2 - 2$$

or

$$du = (3x^2-2)\ dx$$

Looking back at the original problem:

$$\int (3x^2-2)(x^3-2x)^4\ dx = \int (3x^2-2)u^4\ dx = \int u^4\ (3x^2-2)dx$$

Notice, the part "leftover" is du, so substitute:

$$\int u^4\ (3x^2-2)dx = \int u^4\ du$$

Integrate:

$$\int u^4\ du = \frac{u^5}{5} + C$$

Substitute back in $u = x^3 - 2x$ to get

$$\int (3x^2-2)(x^3-2x)^4\ dx = \frac{(x^3-2x)^5}{5} + C$$

- **Example:**

$$\int x^3 \cos(x^4+2)\ dx$$

Let u be the complicated part inside of the cosine function, i.e.,

$$u = x^4 + 2$$

Finding $\frac{du}{dx}$, we get

$$\frac{du}{dx} = 4x^3$$

or

$$du = 4x^3\ dx$$

So, we have:

$$\int x^3 \cos u\ dx = \int \cos u\ x^3 dx$$

Notice,

$$du = 4x^3\ dx$$

We need x^3dx in our problem, so divide both sides of the equation by 4:

$$\frac{1}{4}du = x^3\ dx$$

Now, in place of $x^3\ dx$, substitute $\frac{1}{4}du$:

$$\int \cos u\ x^3 dx = \int \cos u\ \frac{1}{4}du$$

Factoring out the constant, we have

$$\frac{1}{4}\int \cos u\ du = \frac{1}{4}\sin u + C$$

Finally, substituting back in $u = x^4 + 2$, we have the answer:

$$\int x^3 \cos(x^4 + 2)\, dx = \frac{1}{4} \sin(x^4 + 2) + C$$

Group Work:

- **Problems:**
  1. $\int \sqrt{2x+1}\, dx$
  2. $\int \frac{x}{\sqrt{1-4x^2}}\, dx$
  3. $\int e^{5x}\, dx$
- **Answers:**
  1. $\frac{1}{3}(2x+1)^{3/2} + C$
  2. $-\frac{1}{4}\sqrt{1-4x^2} + C$
  3. $\frac{1}{5}e^{5x} + C$

- Working with **definite integrals**, we do the same as above. The biggest difference is that after we get the answer in terms of the original variable, we then evaluate at the integrands.

- **Example:**

$$\int_0^4 \sqrt{2x+1}\, dx$$

Ignore the integrands to begin with and just solve:

$$\int \sqrt{2x+1}\, dx$$

Let

$$u = 2x + 1$$

then

$$\frac{du}{dx} = 2$$

or

$$du = 2\, dx$$

So, we have,

$$\int \sqrt{2x+1}\, dx = \int \sqrt{u}\, dx$$

The only terms left in x is dx, so solve for dx above to get

$$\frac{1}{2} du = dx$$

Substituting in, we have

$$\int \sqrt{u}\, dx = \int \sqrt{u}\, \frac{1}{2} du = \frac{1}{2} \int \sqrt{u}\, du$$

Since,

$$\frac{1}{2} \int \sqrt{u}\, du = \frac{1}{2} \frac{u^{3/2}}{3/2} = \frac{1}{2} \cdot \frac{2}{3} u^{3/2} = \frac{1}{3} u^{3/2}$$

(ignore the C since we are eventually going to find the definite integral), we have (substituting back in $u = 2x + 1$)

$$\int \sqrt{2x+1}\, dx = \frac{1}{3}(2x+1)^{3/2}$$

Now, evaluate

$$\begin{aligned}
\textstyle\int_0^4 \sqrt{2x+1}\, dx &= \tfrac{1}{3}(2x+1)^{3/2}|_0^4 \\
&= \tfrac{1}{3}(2(0)+1)^{3/2} - \tfrac{1}{3}(2(4)+1)^{3/2} \\
&= \tfrac{1}{3} - 9 \\
&= \tfrac{26}{3}
\end{aligned}$$

- **Example:**
$$\int_1^e \frac{\ln x}{x}\ dx$$
First, consider
$$\int \frac{\ln x}{x}\ dx$$
Let
$$u = \ln x$$
since we don't know how to integrate $\ln x$. Then
$$\frac{du}{dx} = \frac{1}{x}$$
or
$$du = \frac{1}{x}\ dx$$
Substituting in u, we have
$$\int \frac{\ln x}{x}\ dx = \int \ln x \frac{1}{x} dx = \int u \frac{1}{x} dx$$
The part leftover is $\frac{1}{x}\ dx$ which is simply du. Thus,
$$\int \frac{\ln x}{x}\ dx = \int u\ du = \frac{u^2}{2}$$
Substituting back in $u = \ln x$, we have:
$$\int \frac{\ln x}{x}\ dx = \frac{u^2}{2} = \frac{(\ln x)^2}{2}$$
Now, we can evaluate at the integrands:
$$\begin{aligned} \textstyle\int_1^e \frac{\ln x}{x}\ dx &= \textstyle\frac{(\ln x)^2}{2}\Big|_1^e \\ &= \textstyle\frac{(\ln e)^2}{2} - \frac{(\ln 1)^2}{2} \\ &= \textstyle\frac{1}{2} \end{aligned}$$

- **Group Work:**
  - **Problems:**
    1. $\int_0^{\sqrt{\pi}} x \cos(x^2)\ dx$
    2. $\int_0^4 \frac{x}{\sqrt{1+2x^2}}\ dx$
  - **Answers:**
    1. 0
    2. 2