Section 3.7
Indeterminant Forms and L'Hopital's Rule
MATH 1190

- We are now going back to limits!!
- Recall
$$\lim_{x\to 1}\frac{x^2-x}{x^2-1}=\lim_{x\to 1}\frac{x(x-1)}{(x-1)(x+1)}=\lim_{x\to 1}\frac{x}{(x+1)}=\frac{1}{1+1}=\frac{1}{2}$$
Note that the original equation
$$\frac{x^2-x}{x^2-1}$$
has the indeterminant form $\frac{0}{0}$ as x goes to 1, so we could factor to get rid of the "problem".
- Look at the following limit
$$\lim_{x\to 1}\frac{\ln x}{x-1}$$
It also has the indeterminant form $\frac{0}{0}$; however, there is no way to factor this to get rid of the "problem". This is where we would use *L'Hopital's Rule.*
- Another indeterminant form:
$$\lim_{x\to\infty}\frac{x^2-1}{2x^2+1}$$
It has the indeterminant form $\frac{\infty}{\infty}$. Previously we solved this problem by dividing by the highest power in the denominator:
$$\lim_{x\to\infty}\frac{x^2-1}{2x^2+1}=\lim_{x\to\infty}\frac{\frac{x^2}{x^2}-\frac{1}{x^2}}{\frac{2x^2}{x^2}+\frac{1}{x^2}}=\lim_{x\to\infty}\frac{1-\frac{1}{x^2}}{2+\frac{1}{x^2}}=\frac{1-0}{2+0}=\frac{1}{2}$$
- **L'Hopital's Rule:** Suppose f and g are differentiable and $g'(x)\neq 0$ near a except possible at a. Suppose that $\frac{f}{g}$ has the indeterminant form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Then
$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$
if the second limit exists.
- Going back to the first example:
$$\lim_{x\to 1}\frac{x^2-x}{x^2-1}$$
Since it has the indeterminant form $\frac{0}{0}$, we can use L'Hopital's rule:
$$\lim_{x\to 1}\frac{x^2-x}{x^2-1}=\lim_{x\to 1}\frac{2x-1}{2x}=\frac{2(1)-1}{2(1)}=\frac{1}{2}$$
which is the same as we found factoring.
- **Example:**
$$\lim_{x\to 1}\frac{\ln x}{x-1}$$
has the indeterminant form $\frac{0}{0}$ as well, so we can use L'Hopital's Rule:
$$\lim_{x\to 1}\frac{\ln x}{x-1}=\lim_{x\to 1}\frac{\frac{1}{x}}{1}=\frac{\frac{1}{1}}{1}=1$$

- **Example:**

$$\lim_{x\to\infty} \frac{e^x}{x^2}$$

has the indeterminant form $\frac{\infty}{\infty}$, so we can use L'Hopital's Rule:

$$\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{e^x}{2x}$$

However,

$$\lim_{x\to\infty} \frac{e^x}{2x}$$

still has the indeterminant form $\frac{\infty}{\infty}$, so we can use L'Hopital's rule again:

$$\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{e^x}{2x} = \lim_{x\to\infty} \frac{e^x}{2} = \frac{\infty}{2} = \infty$$

- **Indeterminant form** $0 \cdot \infty$
  - **Example:**

    $$\lim_{x\to 0^+} x \ln x$$

    has the indeterminant form $0 \cdot \infty$.
  - **Technique:** If $f \cdot g$ has the indeterminant form of $0 \cdot \infty$, rewrite

    $$fg = \frac{f}{\frac{1}{g}} \text{ or } fg = \frac{g}{\frac{1}{f}}$$

    which will result in the indeterminant form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$ which we can then use L'Hopital's rule on.
  - Back to example:
    Note, choosing which function stays in the numerator is important. For example

    $$\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{x}{\frac{1}{\ln x}}$$

    has the indeterminant form of $\frac{0}{0}$ so we can use L'Hopital's rule. However, it is very complicated:

    $$\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{x}{(\ln x)^{-1}} = \lim_{x\to 0^+} \frac{1}{-1(\ln x)^{-2}\frac{1}{x}}$$

    It gets VERY complicated, so we should choose to leave $\ln x$ in the numerator instead!

    $$\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}}$$

    which has the form $\frac{\infty}{\infty}$ which we can still use L'Hopital's rule on:

    $$\begin{aligned}
    \lim_{x\to 0^+} x \ln x &= \lim_{x\to 0^+} \tfrac{\ln x}{\frac{1}{x}} \\
    &= \lim_{x\to 0^+} \tfrac{\ln x}{x^{-1}} && \text{rewriting } \tfrac{1}{x} \\
    &= \lim_{x\to 0^+} \tfrac{\frac{1}{x}}{-x^{-2}} && \text{using L'Hopital's rule} \\
    &= \lim_{x\to 0^+} \tfrac{1}{x} \div \tfrac{-1}{x^2} && \text{rewriting} \\
    &= \lim_{x\to 0^+} \tfrac{1}{x} \cdot \tfrac{-x^2}{1} && \text{multiplying by reciprical} \\
    &= \lim_{x\to 0^+} -x && \text{simplifying} \\
    &= 0 && \text{taking limit}
    \end{aligned}$$

- **Indeterminant form $\infty - \infty$**
  - **Example:**
  $$\lim_{x\to\frac{\pi}{2}^-} (\sec x - \tan x)$$
  has the indeterminant form $\infty - \infty$.
  - **Technique:** In order to use L'Hopital's rule, we need a fraction which has the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Therefore, we need to either write with a common denominator, rationalize or factor out a common factor to get the problem in the appropriate form for using L'Hopital's rule.
  - Back to example:
  $$\lim_{x\to\frac{\pi}{2}^-} (\sec x - \tan x)$$
  We can write out the trigonometric functions in their fraction form and then get a common denominator:
  $$\begin{aligned}\lim_{x\to\frac{\pi}{2}^-} (\sec x - \tan x) &= \lim_{x\to\frac{\pi}{2}^-} \left(\tfrac{1}{\cos x} - \tfrac{\sin x}{\cos x}\right) && \text{rewriting trig functions}\\ &= \lim_{x\to\frac{\pi}{2}^-} \left(\tfrac{1-\sin x}{\cos x}\right) && \text{writing as one fraction}\end{aligned}$$
  Now,
  $$\lim_{x\to\frac{\pi}{2}^-} \left(\frac{1-\sin x}{\cos x}\right)$$
  has the indeterminant form $\frac{0}{0}$ so we can use L'Hopital's rule.
  $$\begin{aligned}\lim_{x\to\frac{\pi}{2}^-} (\sec x - \tan x) &= \lim_{x\to\frac{\pi}{2}^-} \left(\tfrac{1-\sin x}{\cos x}\right) && \text{from above}\\ &= \lim_{x\to\frac{\pi}{2}^-} \tfrac{-\cos x}{-\sin x} && \text{using L'Hopital's rule}\\ &= \tfrac{0}{-1} = 0 && \text{evaluating}\end{aligned}$$
- **Indeterminant Powers 0^0, ∞^0, 1^∞**
  - **Example:**
  $$\lim_{x\to 0}(1+\sin(4x))^{\cot x}$$
  has the indeterminant power 1^∞.
  - **Technique:** If we want to find
  $$\lim_{x\to a} y$$
  and $y = (f(x))^{g(x)}$ has an indeterminant power of 0^0, ∞^0, or 1^∞, then we are first going to find
  $$\lim_{x\to a} \ln y$$
  where $\ln y = \ln (f(x))^{g(x)} = g(x)\cdot \ln f(x)$.
  If
  $$\lim_{x\to a} \ln y = c,$$
  then
  $$\lim_{x\to a} y = e^c.$$
  - Back to example:
  $$\lim_{x\to 0}(1+\sin(4x))^{\cot x}$$
  So
  $$y = (1+\sin(4x))^{\cot x}.$$
  Then
  $$\ln y = \ln\left((1+\sin(4x))^{\cot x}\right) = \cot x \ln(1+\sin(4x)).$$

Looking at $\lim_{x \to a} \ln y$ we have

$$\lim_{x \to 0} \cot x \ln(1 + \sin(4x))$$

which has the indeterminant form $\infty \cdot 0$. We need to rewrite $fg = \frac{f}{\frac{1}{g}}$ or $fg = \frac{g}{\frac{1}{f}}$. Rule of thumb: normally keep the ln in the numerator.

$$\begin{aligned}
\lim_{x \to 0} \cot x \ln(1 + \sin(4x)) &= \lim_{x \to 0} \frac{\ln(1+\sin(4x))}{\frac{1}{\cot x}} && \text{rewriting } fg = \frac{g}{\frac{1}{f}} \\
&= \lim_{x \to 0} \frac{\ln(1+\sin(4x))}{\tan x} && \text{rewriting } \frac{1}{\cot x} = \tan x \\
&= \lim_{x \to 0} \frac{\frac{4\cos(4x)}{(1+\sin(4x))}}{\sec^2 x} && \text{using L'Hopital's rule} \\
&= \frac{\frac{4}{1+0}}{1} = 4 && \text{evaluating}
\end{aligned}$$

Thus,

$$\lim_{x \to 0} (1 + \sin(4x))^{\cot x} = e^4$$

- **Group Work:**
  - **Problems:** Find the following limits
    1. $\lim_{x \to \infty} \frac{\ln x}{\sqrt[3]{x}}$
    2. $\lim_{x \to 0} \frac{\tan x - x}{x^3}$
    3. $\lim_{x \to \pi^-} \frac{\sin x}{1 - \cos x}$
    4. $\lim_{x \to 0} x^x$
    5. $\lim_{x \to 0^+} \ln x - \ln(\sin x)$
    6. $\lim_{x \to \infty} x^{\frac{1}{\ln x}}$
  - **Answers:**
    1. 0
    2. $\frac{1}{3}$
    3. 0
    4. 1
    5. 0
    6. e