Sections 1.4 and 1.6

Limits

MATH 1190

- Consider the function

$$f(x) = \frac{x^2 - 4}{x - 2}$$

It is undefined at $x = 2$, because the denominator equals 0 at $x = 2$.

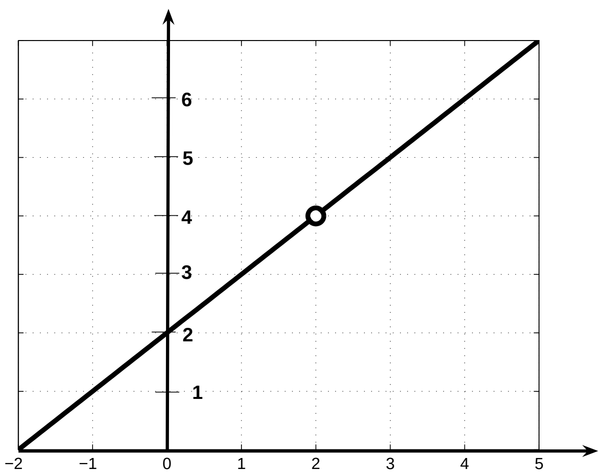
- **Review of Domains:** The domain of a function is normally all real numbers except:
  1. Those values of x which make the denominator equal to 0 (can't have 0 in the denominator).
  2. Those values of x which cause a negative under and even indexed radical.
  3. Those values of x which cause a 0 or negative numbers inside of a logarithm.

- Back to problem:

$$f(x) = \frac{x^2 - 4}{x - 2}$$

Limits have determine the *behavior* of $f(x)$ as x **approaches** some number. In other words, in terms of this problem, what happens to $f(x)$ (*the y-values*) as x gets close to 2 (but *not equal* to 2). What happens as x approaches 2? Graph of

$$f(x) = \frac{x^2 - 4}{x - 2}$$



  - As x gets closer and closer to 2 from the left, what do the y values approach? 4
  - As x gets closer and closer to 2 from the right, what do the y values approach? 4
  - As x gets closer and closer to 2, what do the y values approach? 4

You can also examine a Table of Values, we plug in x values that are approaching 2 into our function $f(x)$, and see what happens to the y values.

  - As x approaches 2 from the left.

x	$f(x)$
1	3
1.9	3.9
1.99	3.99
1.999	3.999

– As x approaches 2 from the right.

x	$f(x)$
3	5
2.1	4.1
2.01	4.01
2.001	4.001

Notice that in both tables, as x gets closer to 2, the y values approach 4. We say, "the limit of $f(x) = \frac{x^2-4}{x-2}$, as x approaches 2, is 4."

- Mathematical notation:

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = 4$$

- **Another example:** Use a table of values to find

$$\lim_{x\to 1} \frac{x^2-1}{x-1}$$

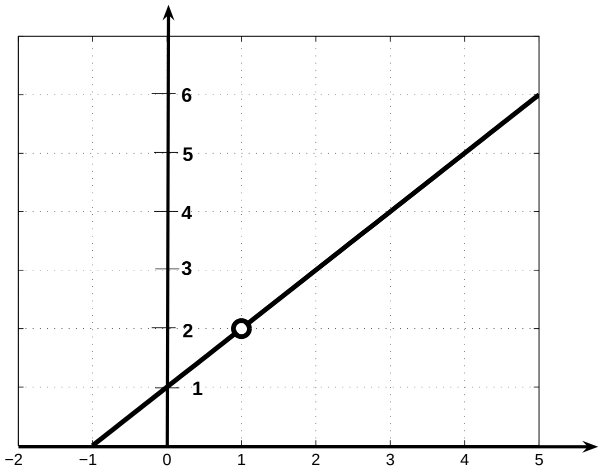
– As x approaches 1 from the left.

x	$f(x)$
0	1
0.9	1.9
0.99	1.99
0.999	1.999

– As x approaches 1 from the right.

x	$f(x)$
2	3
1.1	2.1
1.01	2.01
1.001	2.001

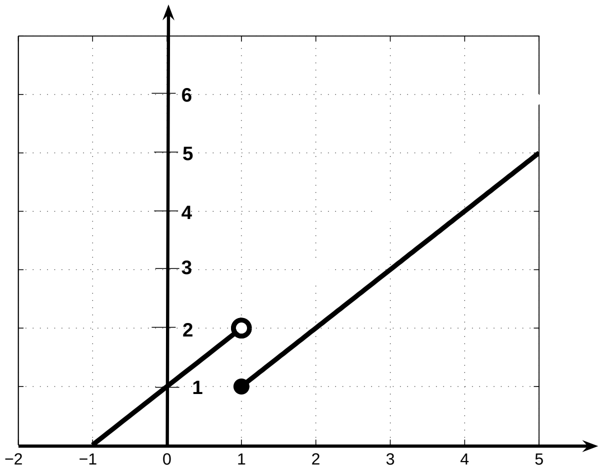
Graphically:



We say

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = 2$$

- **One-handed limits:** Consider the following function:



  - **Left-hand limit:** When you see the *minus* as the exponent of the number, it means what is the limit of $f(x)$ as x approaches the number from the *left*. In the example above, we can examine $f(x)$ as x approaches 1 from the left:

$$\lim_{x \to 1^-} f(x) = 2$$

  - **Right-hand limit:** When you see the *plus* as the exponent of the number, it means what is the limit of $f(x)$ as x approaches the number from the *right*. In the example above, we can examin $f(x)$ as x approaches 1 from the right:

$$\lim_{x \to 1^+} f(x) = 1$$

- **General Definition of Limit** For any function f

$$\lim_{x \to c} f(x) = L$$

  means that as x gets closer and closer to c, but **not equal** to c, from *both* the left and the right, f gets closer and closer to L.

  Note: If

$$\lim_{x \to c^-} f(x) \neq \lim_{x \to c^+} f(x),$$

  then we say *the limit does not exist!*

  Note: $f(c)$ doe NOT need to be defined!

- See plot from class.
- **Group Work - worksheet**
- **Limit Theorems:**

  1. $\lim_{x \to c} k = k$ (The limit of a constant is that constant)
     - Example: $\lim_{x \to 100} \frac{1}{5} = \frac{1}{5}$
  2. $\lim_{x \to c} x = c$ (The limit of x as x approaches c is c)
     - Example: $\lim_{x \to 45} x = 45$

3. *Constant Multiple Rule* $\lim_{x \to c} k \cdot f(x) = k \cdot \lim_{x \to c} f(x)$ (The constant can be taken out of the limit since it doesn't depend on x)
   - Example: $\lim_{x \to 2} 4x = 4 \lim_{x \to 2} x = 4(2) = 8$
4. *Sum Rule* $\lim_{x \to c} [f(x) \pm g(x)] = \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x)$
   - Example: $\lim_{x \to 5} (x+4) = \lim_{x \to 5} x + \lim_{x \to 5} 4 = 5 + 4 = 9$
5. *Product Rule* $\lim_{x \to c} [f(x) \cdot g(x)] = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x)$
   - Example: $\lim_{x \to 5} x(x+4) = \left(\lim_{x \to 5} x\right) \cdot \left(\lim_{x \to 5} (x+4)\right) = 5(9) = 45$
6. *Quotient Rule* $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$ provided $\lim_{x \to c} g(x) \neq 0$.
   - Example: $\lim_{x \to 5} \frac{x+4}{x} = \frac{\lim_{x \to 2} (x+4)}{\lim_{x \to 5} x} = \frac{9}{5}$
7. *Power Rule* $\lim_{x \to c} [f(x)]^n = \left[\lim_{x \to c} f(x)\right]^n$. (You can bring the limit inside)
   - Example: $\lim_{x \to 2} x^3 = \left[\lim_{x \to 2} x\right]^3 = 2^3 = 8$
8. *Simplified Power Rule* $\lim_{x \to c} x^n = c^n$
9. *Root Rule* $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to c} f(x)}$. (You can bring the limit inside)
   - Example: $\lim_{x \to 18} \sqrt{x-2} = \sqrt{\lim_{x \to 18} (x-2)} = \sqrt{18-2} = 4$
10. *Simplified Root Rule* $\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}$.
    - Example: $\lim_{x \to 4} \sqrt{x} = \sqrt{4} = 2$

- All of the Limit Theorems lead us to the **Substitution Property:**

  If f is a continuous function and c is in the domain of f, then

$$\lim_{x \to c} f(x) = f(c)$$

- Example:

$$\lim_{x \to 2} \frac{x^2+3}{5-x} = \frac{2^2+3}{5-2} = \frac{7}{3}$$

- Example:

$$\lim_{x \to \frac{\pi}{2}} (x + \sin x) = \frac{\pi}{2} + \sin\left(\frac{\pi}{2}\right) = \frac{\pi}{2} + 1$$

- **Indeterminant Form:** $\frac{0}{0}$ Let's look back at the original example

$$\lim_{x \to 2} \frac{x^2-4}{x-2}$$

  Notice that we cannot use the Substitution Property, because 2 is NOT in the domain of our function, because if you plug in 2 to the denominator, you get 0. However, if you plug 2 into the numerator, you also get 0. This is the indeterminant form $\frac{0}{0}$. In this case, there is a way to "get rid of the problem".

- In our example, we can factor the numerator and cancel out the "problem".

$$\lim_{x \to 2} \frac{x^2-4}{x-2} = \lim_{x \to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x \to 2} (x+2) = 4$$

- Example:

$$\lim_{x\to 3} \frac{x-3}{x^2-x-6} = \lim_{x\to 3} \frac{x-3}{(x-3)(x+2)} = \frac{1}{5}$$

- Example:

$$\begin{aligned}
\lim_{h\to 0} \tfrac{(3+h)^2-9}{h} &= \lim_{h\to 0} \tfrac{(3+h)(3+h)-9}{h} \\
&= \lim_{h\to 0} \tfrac{9+6h+h^2-9}{h} \\
&= \lim_{h\to 0} \tfrac{6h+h^2}{h} \\
&= \lim_{h\to 0} \tfrac{h(6+h)}{h} \\
&= \lim_{h\to 0} (6+h) \\
&= 6+0 \\
&= 6
\end{aligned}$$

- Example:

$$\lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2}$$

Notice, this still has the form $\frac{0}{0}$, but we can't factor. In this case we need to rationalize:

$$\begin{aligned}
\lim_{t\to 0} \tfrac{\sqrt{t^2+9}-3}{t^2} &= \lim_{t\to 0} \tfrac{\sqrt{t^2+9}-3}{t^2} \cdot \tfrac{\sqrt{t^2+9}+3}{\sqrt{t^2+9}+3} \\
&= \lim_{t\to 0} \tfrac{t^2+9-9}{t^2(\sqrt{t^2+9}+3)} \\
&= \lim_{t\to 0} \tfrac{t^2}{t^2(\sqrt{t^2+9}+3)} \\
&= \lim_{t\to 0} \tfrac{1}{\sqrt{t^2+9}+3} \\
&= \tfrac{1}{\sqrt{0^2+9}+3} \\
&= \tfrac{1}{6}
\end{aligned}$$

- **Group Work - worksheet**