Section 1.8
Limits Involving Infinity
MATH 1190

- **Limits that go to infinity:**
  - Suppose we have a rational function that has a zero denominator when we are trying to substitute to find the limit. Sometimes we have the form $\frac{0}{0}$ and we can either factor or rationalize.
  - What about
  $$\lim_{x\to 0}\frac{1}{x^2}?$$
  The denominator goes to 0, but the numerator does not. Look at a table:
    * As x approaches 0 from the left.

x	$f(x)$
-1	1
-0.1	100
-0.01	10,000
-0.001	1,000,000

    * As x approaches 0 from the right.

x	$f(x)$
1	1
0.1	100
0.01	10,000
0.001	1,000,000

  As x goes to 0 from both the left and right $\frac{1}{x^2}$ grows larger and larger in the positive direction, i.e., it is approaching $+\infty$
  - *Notation:*
  $$\lim_{x\to a} f(x) = \infty$$
  means that as the values of x get closer and closer to a, $f(x)$ grows without bound or gets arbitrarily large.
  - In our example,
  $$\lim_{x\to 0}\frac{1}{x^2} = \infty$$
  Also,
  $$\lim_{x\to 0^-}\frac{1}{x^2} = \infty$$
  and
  $$\lim_{x\to 0^+}\frac{1}{x^2} = \infty$$
  - Example: What are
  $$\lim_{x\to 0^-}\frac{1}{x}$$
  and
  $$\lim_{x\to 0^+}\frac{1}{x}$$
  We know that in both of these, as x gets really, really small (close to 0), the entire fraction is going to get really, really large and go to either positive or negative infinity. The problem is to figure out which.
  As $x \to 0^-$, the values of x will be negative (because from the left), so $\frac{1}{x}$ will have a positive number in the numerator, and a negative number in the denominator, so the entire fraction is getting larger in the negative direction. Thus,
  $$\lim_{x\to 0^-}\frac{1}{x} = -\infty$$

Similarly, as $x \to 0^+$, the values of x will be positive (because from the right), so $\frac{1}{x}$ will have a positive number in the numerator, and a positive number in the denominator, so the entire fraction is getting larger in the positive direction. Thus,

$$\lim_{x \to 0^+} \frac{1}{x} = +\infty$$

As a result,

$$\lim_{x \to 0} \frac{1}{x} = DNE \text{ (does not exist)}$$

- Example: Find

$$\lim_{x \to 3^-} \frac{2x}{3-x}, \quad \lim_{x \to 3^+} \frac{2x}{3-x}, \quad \lim_{x \to 3} \frac{2x}{3-x}$$

- Example:

$$\lim_{x \to 2^-} \left(\frac{1}{x-2} + 3 \right)$$

Notice, we can write:

$$\begin{aligned} \lim_{x \to 2^-} \left(\tfrac{1}{x-2} + 3 \right) &= \lim_{x \to 2^-} \tfrac{1}{x-2} + \lim_{x \to 2^-} 3 \\ &= -\infty + 3 \\ &= -\infty \end{aligned}$$

Similarly,

$$\lim_{x \to 2^+} \left(\frac{1}{x-2} + 3 \right) = +\infty + 3 = +\infty$$

- **Limits at infinity:**
  - Now we are looking at what happens as x goes to infinity. Look at

$$\lim_{x \to \infty} \frac{1}{x}$$

x	$f(x)$
1	1
10	0.1
100	0.01
1000	0.001

We have

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

Similarly,

$$\lim_{x \to \infty} \frac{1}{x^n} = 0$$

regardless of the value of n.

– Example: Find

$$\lim_{x\to\infty} \frac{3x-5}{x-2}$$

Steps:

1. Divide every term by the highest power in the *denominator*:

$$\lim_{x\to\infty} \frac{3x-5}{x-2} = \lim_{x\to\infty} \frac{\frac{3x}{x} - \frac{5}{x}}{\frac{x}{x} - \frac{2}{x}}$$

2. Simplify:

$$\lim_{x\to\infty} \frac{\frac{3x}{x} - \frac{5}{x}}{\frac{x}{x} - \frac{2}{x}} = \lim_{x\to\infty} \frac{3 - \frac{5}{x}}{1 - \frac{2}{x}}$$

3. Use the fact that $\lim\limits_{x\to\infty} \frac{1}{x^n} = 0$ to find the answer.

$$\lim_{x\to\infty} \frac{3 - \frac{5}{x}}{1 - \frac{2}{x}} = \frac{3-0}{1-0} = 3$$

– Example:

$$\lim_{x\to\infty} \frac{x^2-x}{x^2+1} = \lim_{x\to\infty} \frac{\frac{x^2}{x^2} - \frac{x}{x^2}}{\frac{x^2}{x^2} + \frac{1}{x^2}} = \lim_{x\to\infty} \frac{1-\frac{1}{x}}{1+\frac{1}{x^2}} = 1$$

– Example:

$$\begin{aligned} \lim_{x\to-\infty} \tfrac{x^{42}-16x+5}{x^{43}+12x^{42}} &= \lim_{x\to-\infty} \tfrac{\frac{x^{42}}{x^{43}} - \frac{16x}{x^{43}} + \frac{5}{x^{43}}}{\frac{x^{43}}{x^{43}} + \frac{12x^{42}}{x^{43}}} \\ &= \lim_{x\to-\infty} \tfrac{\frac{1}{x} - \frac{16}{x^{42}} + \frac{5}{x^{43}}}{1+\frac{12}{x}} \\ &= \tfrac{0-0+0}{1+0} \\ &= 0 \end{aligned}$$

– Example:

$$\begin{aligned} \lim_{x\to-\infty} \tfrac{x^9-16x+5}{x^4+12x^8} &= \lim_{x\to-\infty} \tfrac{\frac{x^9}{x^8} - \frac{16x}{x^8} + \frac{5}{x^8}}{\frac{x^4}{x^8} + \frac{12x^8}{x^8}} \\ &= \lim_{x\to-\infty} \tfrac{x - \frac{16}{x^7} + \frac{5}{x^8}}{\frac{1}{x^4}+12} \\ &= \lim_{x\to-\infty} \tfrac{x-0+0}{0+12} \\ &= -\infty \end{aligned}$$

- **Group Work - worksheet**