Section 2.10 Part 2
Logarithmic Differentiation
MATH 1190

- Logarithmic differentiation is used in two instances.
  1. When there is an extremely complicated function which would require numerous rules. For example:
  $$y = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$$
  Notice with this function, we would have to use the quotient rule, product rule, chain rule and the normal power rule all on one problem. On problems like these, logarithmic differentiation simply makes the problem easier.
  2. The second case in which you use logarithmic differentiation is when you have a function of the form $y = (f(x))^{g(x)}$. In this case, logarithmic differentiation is the **only** way to find the derivative. An example of this type of function is:
  $$y = (3x^2)^{\sin x}$$
- Looking at the first problem, we will walk through the steps necessary for logarithmic differentiation.
$$y = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$$
  - **Step 1:** Take the ln of both sides:
  $$\ln y = \ln\left(\frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\right)$$
  - **Step 2:** Use the rules of logarithms to simplify:
  $$\begin{aligned}
  \ln y &= \ln\left(\tfrac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\right) \\
  &= \ln\left[x^{3/4}(x^2+1)^{1/2}\right] - \ln\left[(3x+2)^5\right] && \text{using quotient rule of logarithms} \\
  &= \ln x^{3/4} + \ln(x^2+1)^{1/2} - \ln\left[(3x+2)^5\right] && \text{using product rule of logarithms} \\
  &= \tfrac{3}{4}\ln x + \tfrac{1}{2}\ln(x^2+1) - 5\ln(3x+2) && \text{using power rule of logarithms}
  \end{aligned}$$
  - **Step 3:** Now use implicit differentiation to find $\frac{dy}{dx}$. One thing to note - the left hand side will ALWAYS be $\frac{1}{y}\frac{dy}{dx}$, AND this will be the only term containing $\frac{dy}{dx}$.
  $$\begin{aligned}
  \tfrac{d}{dx}y &= \tfrac{d}{dx}\left(\tfrac{3}{4}\ln x + \tfrac{1}{2}\ln(x^2+1) - 5\ln(3x+2)\right) && \text{taking the derivative of both sides w.r.t. } x \\
  \tfrac{1}{y}\tfrac{dy}{dx} &= \tfrac{3}{4}\tfrac{1}{x} + \tfrac{1}{2}\tfrac{1}{x^2+1}\tfrac{d}{dx}(x^2+1) - 5\tfrac{1}{3x+2}\tfrac{d}{dx}(3x+2) && \text{using the differentiation rules for logarithms w/ and w/out the chain rule} \\
  \tfrac{1}{y}\tfrac{dy}{dx} &= \tfrac{3}{4}\tfrac{1}{x} + \tfrac{1}{2}\tfrac{1}{x^2+1}(2x) - 5\tfrac{1}{3x+2}(3) && \text{finishing taking derivatives} \\
  \tfrac{1}{y}\tfrac{dy}{dx} &= \tfrac{3}{4x} + \tfrac{x}{x^2+1} - \tfrac{15}{3x+2} && \text{simplifying} \\
  \tfrac{dy}{dx} &= y\left(\tfrac{3}{4x} + \tfrac{x}{x^2+1} - \tfrac{15}{3x+2}\right) && \text{getting } \tfrac{dy}{dx} \text{ by itself}
  \end{aligned}$$

- **Step 4:** Since the function started with all x's, we must end with all x's in the answer. Therefore, the last step is to substitute back in the original equation for y.

$$\frac{dy}{dx} = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

- The next group of problems are ones in which we have $y = (f(x))^{g(x)}$. Before going through an example for this, let's look at the difference between this type of problem and those we have already examined.
  - An example of a problem we have already examined: if we have the function $y = e^{\sin x}$, notice that we have a function of x in the exponent, $(\sin x)$, but the base is a constant, (e). Therefore, we would take the derivative using the rule for e^x along with the chain rule to get:

    $$\frac{dy}{dx} = e^{\sin x}\frac{d}{dx}(\sin x) = e^{\sin x}\cos x$$

    This does *not* require logarithmic differentiation.
  - Another example of a problem we have already examined: $y = 3^{-x}$. Again, the base is a constant, (3), and the exponent is a function of x, $(-x)$. To get the derivative, use $\frac{d}{dx}(a^u) = a^u \ln(a)\frac{du}{dx}$. In this problem you will get

    $$y' = 3^{-x}\ln(3)(-1) = -3^{-x}\ln(3)$$

    This again does *not* require logarithmic differentiation.
  - One more example of a problem we already of done: $f(x) = (\sin x)^6$. In this case there is a function of x, $(\sin x)$, raised to a constant power, (6). Therefore, we use the normal techniques for chain rule where we have something complicated raised to a power.

    $$f'(x) = 6(\sin x)^5\frac{d}{dx}(\sin x) = 6(\sin x)^5\cos x = 6\cos x(\sin x)^5$$

    This again does *not* require logarithmic differentiation.

The above problems we have already examined in previous sections. However, supposed you have the function

$$y = (3x^2)^{\sin x}$$

where you have a function of x, $(3x^2)$, raised to a function of x, $(\sin x)$. Another example of a function in which you would need to use logarithmic differentiation is

$$y = (\ln x)^{\ln x}$$

where again you have a function of x, $(\ln x)$, raised to a function of x, $(\ln x)$. In both of these examples, you need to use logarithmic differentiation, there is NO other way to find the derivative. Let's look at each of these examples in detail:

1. **Example:** $y = (3x^2)^{\sin x}$; Let's go through the same steps of logarithmic differentiation for this problem:
   - **Step 1:** Take the ln of both sides:

     $$\ln y = \ln\left[(3x^2)^{\sin x}\right]$$

   - **Step 2:** Use the rules of logarithms to simplify:

     $$\begin{aligned}\ln y &= \ln\left[(3x^2)^{\sin x}\right] \\ &= \sin x\left(\ln(3x^2)\right) \qquad \text{using the power rule}\end{aligned}$$

- **Step 3:** Now use implicit differentiation to find $\frac{dy}{dx}$. Remember, the left hand side will ALWAYS be $\frac{1}{y}\frac{dy}{dx}$.

$$\begin{aligned}
\frac{1}{y}\frac{dy}{dx} &= \sin x \frac{d}{dx}\left(\ln\left(3x^2\right)\right) + \ln\left(3x^2\right)\frac{d}{dx}(\sin x) && \text{using the product rule} \\
\frac{1}{y}\frac{dy}{dx} &= \sin x \frac{1}{3x^2}\frac{d}{dx}(3x^2) + \ln\left(3x^2\right)\cos x && \text{using the chain rule on the first term and } \frac{d}{dx}\sin x = \\
\frac{1}{y}\frac{dy}{dx} &= \sin x \frac{1}{3x^2}(6x) + \ln\left(3x^2\right)\cos x && \text{finishing taking the derivative} \\
\frac{1}{y}\frac{dy}{dx} &= \frac{2\sin x}{x} + \cos x \ln\left(3x^2\right) && \text{simplifying} \\
\frac{dy}{dx} &= y\left(\frac{2\sin x}{x} + \cos x \ln\left(3x^2\right)\right) && \text{solving for } \frac{dy}{dx}
\end{aligned}$$

- **Step 4:** Substitute back in the original equation for y.

$$\frac{dy}{dx} = \left(3x^2\right)^{\sin x}\left(\frac{2\sin x}{x} + \cos x \ln\left(3x^2\right)\right)$$

2. **Example:** $y = (\ln x)^{\ln x}$
   - **Step 1:** Take the ln of both sides:

$$\ln y = \ln\left((\ln x)^{\ln x}\right)$$

   - **Step 2:** Use the rules of logarithms to simplify:

$$\begin{aligned}
\ln y &= \ln\left((\ln x)^{\ln x}\right) \\
&= \ln x\left(\ln\left(\ln x\right)\right) && \text{using the power rule}
\end{aligned}$$

   - **Step 3:** Now use implicit differentiation to find $\frac{dy}{dx}$. Remember, the left hand side will ALWAYS be $\frac{1}{y}\frac{dy}{dx}$.

$$\begin{aligned}
\frac{1}{y}\frac{dy}{dx} &= \ln x \frac{d}{dx}\left(\ln\left(\ln x\right)\right) + \ln\left(\ln x\right)\frac{d}{dx}(\ln x) && \text{using the product rule} \\
\frac{1}{y}\frac{dy}{dx} &= \ln x \frac{1}{\ln x}\frac{d}{dx}(\ln x) + \ln\left(\ln x\right)\frac{1}{x} && \text{using the chain rule on the first term and } \frac{d}{dx}\ln x = \frac{1}{x} \\
\frac{1}{y}\frac{dy}{dx} &= \ln x \frac{1}{\ln x}\left(\frac{1}{x}\right) + \ln\left(\ln x\right)\frac{1}{x} && \text{finishing taking the derivative} \\
\frac{1}{y}\frac{dy}{dx} &= \frac{\ln x}{x\ln x} + \frac{\ln\left(\ln x\right)}{x} && \text{simplifying} \\
\frac{dy}{dx} &= y\left(\frac{\ln x}{x\ln x} + \frac{\ln\left(\ln x\right)}{x}\right) && \text{solving for } \frac{dy}{dx}
\end{aligned}$$

   - **Step 4:** Substitute back in the original equation for y.

$$\frac{dy}{dx} = (\ln x)^{\ln x}\left(\frac{\ln x}{x\ln x} + \frac{\ln\left(\ln x\right)}{x}\right)$$

- **Group Work:** Work on these problems in teams of 2-3. Use logarithmic differentiation to find the derivative of the following functions
  - Problems
    1. $y = \frac{\sqrt{x}e^{x^2}(x^2+1)^{10}}{(2x+1)^5}$
    2. $y = \sqrt{x}^x$
  - Answers
    1. $\frac{dy}{dx} = \frac{\sqrt{x}e^{x^2}(x^2+1)^{10}}{(2x+1)^5}\left(\frac{1}{2x} + 2x + \frac{20x}{x^2+1} - \frac{10}{2x+1}\right)$
    2. $\frac{dy}{dx} = \frac{\sqrt{x}^x}{2}\left(1 + \ln x\right)$