Section 2.10 Part 1
Logarithmic Functions
MATH 1190

- **Review from PreCalculus:**
  - A logarithmic function has the form
    $$y = \log_a x$$
    It's equivalent exponential form is
    $$a^y = x$$
    where a is the base, $a > 0$, $a \neq 1$.
  - Special bases:
    * Base 10:
      $$y = \log x$$
    * Base e:
      $$y = \ln x$$
      Note: $\ln(e) = 1$ and $e^{\ln x} = x$
  - In general
    $$a^{\log_a x} = x$$
  - Properties:
    1. Product Rule: $\log_a(xy) = \log_a x + \log_a y$
    2. Quotient Rule: $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$
    3. Reciprocal Rule: $\log_a\left(\frac{1}{y}\right) = -\log_a y$
    4. Power Rule: $log_a(x^p) = p\log_a x$

- **Calculus Material:**
  - **Derivative of a^x**
    Recall:
    $$\begin{aligned} a^x &= e^{\ln(a^x)} && \text{b/c } e^{\ln x} = x \\ &= e^{x \ln a} && \text{using the power rule} \end{aligned}$$
    So,
    $$\begin{aligned} \frac{d}{dx}(a^x) &= \frac{d}{dx}(e^{x\ln(a)}) && \text{b/c } a^x = e^{x\ln a} \\ &= e^{x\ln(a)}\frac{d}{dx}(x\ln(a)) && \text{using the chain rule} \\ &= e^{x\ln(a)}(\ln(a)) && \text{b/c } \tfrac{d}{dx}(cx) = c \text{ for any constant } c \\ & && \text{and } \ln(a) \text{is just a constant} \\ &= \ln(a)e^{x\ln(a)} && \text{rearranging terms} \\ &= \ln(a)e^{\ln(a^x)} && \text{using the power rule} \\ &= \ln(a)a^x && \text{b/c } e^{\ln x} = x \end{aligned}$$
    Therefore, we have
    $$\boxed{\frac{d}{dx}a^x = a^x \ln(a)}$$

Similarly,using the chain rule, we also have

$$\boxed{\frac{d}{dx}a^u = a^u \ln(a)\frac{du}{dx}}$$

* **Example:** Let $y = 6^{\sin x}$. Find y'.

$$y' = 6^{\sin x}\ln(6)\frac{d}{dx}(\sin x) = 6^{\sin x}\ln(6)\cos x$$

* **Example:** Let $f(s) = 2^{s^2}$. Find $f'(s)$

$$f'(s) = 2^{s^2}\ln(2)\frac{d}{ds}(s^2) = 2^{s^2}\ln(2)(2s)$$

– **Derivative of $\ln x$:**

$$\boxed{\frac{d}{dx}\ln x = \frac{1}{x}}$$

* **Example:** $y = x^6\ln x - \frac{1}{4}x^4$. Find y'.

$$\begin{aligned} y' &= x^6\frac{d}{dx}(\ln x) + \ln x\frac{d}{dx}(x^6) - x^3 && \text{using the product rule on the first term} \\ &= x^6\left(\frac{1}{x}\right) + \ln x(6x^5) - x^3 && \text{b/c } \frac{d}{dx}(\ln x) = \frac{1}{x} \\ &= x^5 + 6x^5\ln x - x^3 && \text{simplifying} \end{aligned}$$

* **Example:** Let $y = \ln(3x)$, find y'.
Notice on this example, that there is something more complicated inside the natural logarithm function than just an x. Since there is something more complicated, we must use the chain rule. For logarithmic functions, if we use the chain rule, we have

$$\boxed{\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx} = \frac{\frac{du}{dx}}{u}}$$

So, in this example, we will have

$$y' = \frac{1}{3x}\frac{d}{dx}(3x) = \frac{1}{3x}(3) = \frac{1}{x}$$

– **Derivative of $\log_a x$**
In order to derive this derivative, we must first remember the change of base formula from Pre-Calculus:

$$\log_a x = \frac{\ln x}{\ln a}$$

Then

$$\begin{aligned} \frac{d}{dx}\log_a x &= \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right) && \text{using the change of base formula above} \\ &= \frac{d}{dx}\left(\frac{1}{\ln a}\ln x\right) && \text{rewriting} \\ &= \frac{1}{\ln a}\frac{d}{dx}(\ln x) && \text{moving the constant in front} \\ &= \frac{1}{\ln a}\frac{1}{x} && \text{b/c } \frac{d}{dx}(\ln x) = \frac{1}{x} \\ &= \frac{1}{x\ln a} && \text{simplifying} \end{aligned}$$

So,

$$\boxed{\frac{d}{dx}\log_a x = \frac{1}{x\ln a}}$$

and

$$\boxed{\frac{d}{dx}\log_a u = \frac{1}{u\ln a}\frac{du}{dx}}$$

using the chain rule.

  * **Example:** Let

$$y = \log_7 \left(\frac{\sin\theta \cos\theta}{e^{\theta} 2^{\theta}} \right).$$

    Find y'. Notice on this problem, it is much easier to simplify the original function first using the properties of logarithms.

$$\begin{aligned} y &= \log_7 \left(\tfrac{\sin\theta\cos\theta}{e^{\theta}2^{\theta}}\right) \\ &= \log_7(\sin\theta\cos\theta) - \log_7(e^{\theta}2^{\theta}) && \text{using quotient rule} \\ &= \log_7(\sin\theta) + \log_7(\cos\theta) - \big(\log_7(e^{\theta}) + \log_7(2^{\theta})\big) && \text{using the product rule} \\ &= \log_7(\sin\theta) + \log_7(\cos\theta) - \log_7(e^{\theta}) - \log_7(2^{\theta}) && \text{distributing the negative sign} \end{aligned}$$

    Now, you are ready to take the derivative of each terms:

$$\begin{aligned} y' &= \tfrac{1}{\sin(\theta)\ln(7)}\tfrac{d}{d\theta}\sin(\theta) + \tfrac{1}{\cos(\theta)\ln(7)}\tfrac{d}{d\theta}\cos(\theta) - \tfrac{1}{e^{\theta}\ln(7)}\tfrac{d}{d\theta}e^{\theta} - \tfrac{1}{2^{\theta}\ln(7)}\tfrac{d}{d\theta}2^{\theta} \\ &= \tfrac{1}{\sin(\theta)\ln(7)}\cos(\theta) + \tfrac{1}{\cos(\theta)\ln(7)}(-\sin(\theta)) - \tfrac{1}{e^{\theta}\ln(7)}e^{\theta} - \tfrac{1}{2^{\theta}\ln(7)}2^{\theta}\ln(2) \\ &= \tfrac{\cos(\theta)}{\sin(\theta)\ln(7)} - \tfrac{\sin(\theta)}{\cos(\theta)\ln(7)} - \tfrac{1}{\ln(7)} - \tfrac{\ln(2)}{\ln(7)} \end{aligned}$$

- **Group Work:** Work on these problems in teams of 3-4. Find the derivative of the following functions.

  * Problems
    1. $y = \ln(t^{3/2})$
    2. $f(x) = x\ln(7x^2 + 5x + 2)$
    3. $y = \ln\sqrt{\frac{(x+1)^5}{(x+2)^{20}}}$
    4. $f(x) = \log_2\left(\frac{\tan x}{\sec x}\right)$
    5. $f(x) = 5^{x^2-2} + \ln 2x + 6e^x$
    6. $y = 3\log_8(\log_2 t)$
    7. $y = [\ln(1 + e^x)]^2$
  * Answers
    1. $y' = \frac{3}{2t}$
    2. $f'(x) = \frac{14x^2+5x}{7x^2+5x+2} + \ln(7x^2 + 5x + 2)$
    3. $y' = \frac{5}{2(x+1)} - \frac{10}{x+2}$
    4. $f'(x) = \frac{\sec^x}{\tan x \ln(2)} - \frac{\tan x}{\ln(2)}$
    5. $f'(x) = 2x\ln(5)5^{x^2-2} + \frac{1}{x} + 6e^x$
    6. $y' = \frac{3}{t\log_2 t\ln(8)\ln(2)}$
    7. $y' = \frac{2e^x\ln(1+e^x)}{1+e^x}$