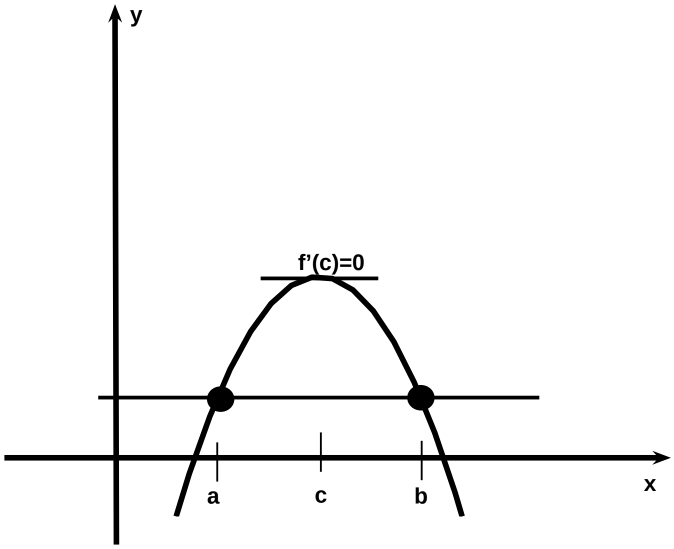
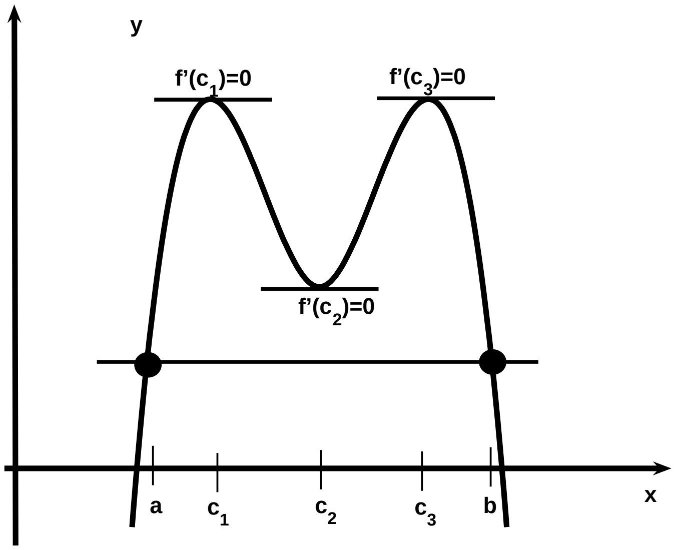
Section 3.2
Mean Value Theorem
MATH 1190

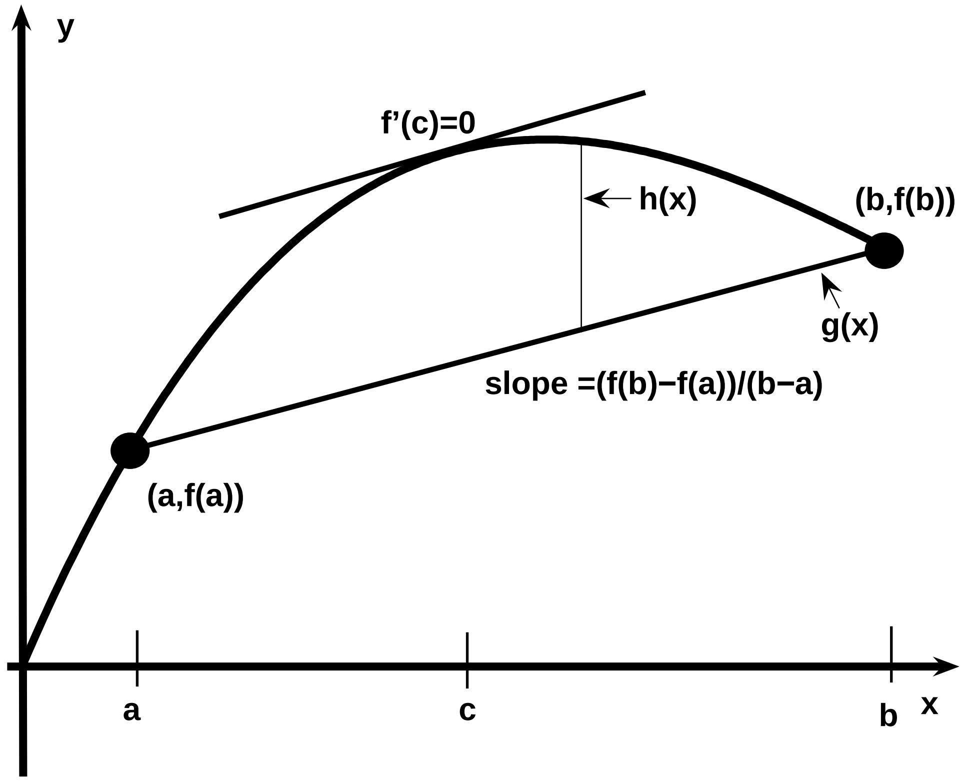
- **Rolle's Theorem:** Suppose $y = f(x)$ is continuous at every point $[a, b]$ and differentiable at every point (a, b). If $f(a) = f(b)$, then there is at least one number c in (a, b) at which $f'(c) = 0$. (see the pictures below.)





- **Mean Value Theorem:** Suppose $y = f(x)$ is continuous on $[a, b]$ and differentiable on (a, b). Then there exists at least one point c in (a, b) at which

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



Proof: Let $g(x)$ be the line passing through $(a, f(a))$ and $(b, f(b))$. Then the line passing through the points can be found using the point slope for where

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(b) - f(a)}{b - a}$$

Thus, using $y - y_1 = m(x - x_1)$, we have

$$y - f(a) = \tfrac{f(b)-f(a)}{b-a}(x - a)$$

$$y = f(a) + \tfrac{f(b)-f(a)}{b-a}(x - a)$$

So,

$$g(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$$

Let $h(x)$ represent the distance between $f(x)$ and $g(x)$ at any point x. In other words,

$$h(x) = f(x) - g(x)$$

Using the formula for $g(x)$, we get

$$h(x) = f(x) - \left(f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \right)$$

Notice that

$$h(a) = f(a) - \left(f(a) + \frac{f(b) - f(a)}{b - a}(a - a) \right) = 0$$

and

$$h(b) = f(b) - \left(f(a) + \frac{f(b) - f(a)}{b - a}(b - a) \right) = 0$$

Using Rolle's theorem, since $h(a) = h(b)$, there exists at least one point c in (a, b) such that $h'(c) = 0$. Differentiating $h(x)$, we have

$$\begin{aligned} h'(x) &= fracddx \left(f(x) - \left(f(a) + \tfrac{f(b)-f(a)}{b-a}(x-a) \right) \right) \\ &= f'(x) - \tfrac{f(b)-f(a)}{b-a} \end{aligned}$$

So,

$$h'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}$$

Thus, since $h'(c) = 0$, we have

$$0 = f'(c) - \frac{f(b) - f(a)}{b - a}$$

or

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

This proves the Mean Value Theorem.

- **Consequences of Mean Value Theorem**

  1. If $f'(x) = 0$ at each point x of an open interval (a, b), then $f(x) = C$ for all x in (a, b), where C is a constant.

     Proof: Let x_1 and x_2 be any two points in the interval (a, b). I'll show that regardless of what x_1 and x_2 are $f(x_1) = f(x_2)$ for any two points x_1 and x_2. This will mean that $f(x)$ is constant. f satisfies the Mean Value Theorem which means that there is a point c in $[x_1, x_2]$ such that

     $$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

     But, since $f'(x) = 0$ for any x, then $f'(c) = 0$. So,

     $$0 = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

     which means that $f(x_2) - f(x_1) = 0$ or $f(x_2) = f(x_1)$. Thus for any two points x_1 and x_2 in (a, b). So, $f(x) = C$ where C is a constant.

  2. If $f'(x) = g'(x)$ at each point x in an open interval (a, b), then there exists a constant C such that $f(x) = g(x) + C$ for all x in (a, b). In other words, $f - g$ is a constant on (a, b).

     Proof: Let $h = f - g$. Then $h'(x) = f'(x) - g'(x) = 0$ since $f'(x) = g'(x)$. From the first consequence, we have $h(x) = C$ where C is a constant. So, $f(x) - g(x) = C$ or $f(x) = g(x) + C$.